Inconsistency in Mathematics and Inconsistency in Chemistry

Michèle Friend[†]
michele@gwu.edu

ABSTRACT

In this paper, I compare how it is that inconsistencies are handled in mathematics to how they are handled in chemistry. In mathematics, they are very precisely formulated and identified, unlike in chemistry. So the chemists can learn from the precision and the very well-worked out strategies developed by logicians and deployed by mathematicians to cope with inconsistency. Some lessons can also be learned by the mathematicians from the chemists. Mathematicians tend to be intolerant towards inconsistencies. There are some philosophers of chemistry who attribute to chemists, collectively, a more tolerant attitude towards inconsistency, and so, mathematicians can learn from the chemists' attitude to inform their choice of strategies.

Introduction

I shall discuss two sorts of mathematical inconsistency: that within a mathematical theory and that between mathematical theories. In order to do this, I shall first have to say how to individuate a theory in mathematics. Taking the various ways of individuating theories into account, we can then say broadly that mathematicians are most intolerant towards inconsistency *within* a theory and more tolerant towards inconsistency *between* theories. I shall give some indication as to how these are handled in mathematics. Having written about inconsistencies in mathematics, I shall then turn to inconsistencies in chemistry.[3]

[†] Department of Philosophy, George Washington University, Washington D.C., U.S.

[3] The reason for choosing chemistry is to take a little distance from the usual "philosophy of science means philosophy of physics". Chemistry is, in some ways more representative of 'science'. There are few laws, it is a very pluralist discipline, and the concept of substance is metaphysically not a neat as object is in physics. That

In chemistry, theories are not so neatly individuated, and therefore, the 'within a theory' / 'between theories' distinction is not relevant. Some philosophers of chemistry, for example, one can read Schummer (2015) and Chang (2012), as implicitly attributing a high level of tolerance towards inconsistency in chemistry, at least collectively if not individually, and at least 'at times' if not indefinitely. The question then arises as to where the inconsistencies in chemistry are located. I shall make a proposal: that inconsistency in chemistry is either located in the chemicals themselves or in how it is that we express and develop our theories about the, then presumably consistent, chemicals and their reactions. The first sort of inconsistency is very hard for 'Western' scientists to even entertain, and I do not pretend to understand it myself, but for argumentative reasons we cannot foreclose on this possibility *a priori*. Therefore, I shall try to make some partial sense of it.

In the conclusion, we shall see how the mathematician can inform the chemist and how the chemist can inform the mathematician. The mathematicians stand to learn tolerance from the chemists, and the chemists stand to learn precision of formulation of concepts and logical tools for coping with inconsistency.

1. Individuating Mathematical Theories[4]

Prima facie,[5] in mathematics, theories can be quite rigidly and precisely delineated, or individuated. That is, we can precisely give their identity conditions: conditions that allow us to tell if two presentations of theories are in fact equivalent or not, and to tell if we have overstepped the bounds of a theory.[6]

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is, substances are not individuated or bounded as clearly as objects are in physics, at least *prima facie*. That is, we might find that the same problems I identify here also apply to physics, but they are clearer in chemistry.

[4] I should like to thank the Instituto de Investigaciones Filosóficas at the Universidad Nacional Autónoma de México for inviting me to two workshops in September 2016 and generously hosting me for two months. It was largely for the workshops and while at the institute that I developed this paper.

[5] We shall qualify this shortly. Look out for the '*secunda facie*'.

[6] Of course, 'equivalence' is always determined in terms of particular features (predicates, relations, functions, operations, cardinality of domains etc.). So, we might think that this is just a relative term, but it is not. In mathematics, we can be completely precise about how we determine equivalence. We can also spot immediately some respects in which two presentations of theories are equivalent but not identical. The gap between equivalence and identity might be deemed relevant or not, and this is how we make the judgment that two theories are equivalent up to our interests or concerns. These details are seldom mentioned, but are always there, and are worth bearing in mind.

The favourite way of individuating theories for philosophers of mathematics in the late twentieth century up to today, is in terms of:

(a)   language, axioms and a syntax,

such as when we talk of Zermelo-Fraenkel set theory, or second-order Peano Arithmetic. This way of individuating theories in mathematics has come to us under the influence of Hilbert and Bourbaki amongst others. The advantage of this type of individuation is its precision. However, not every theory in mathematics is presented axiomatically (for example: model theory or the calculus). Some philosophers might think that, nevertheless, every theory of mathematics *can in principle* be so presented, but this is only the case if we mean this *a priori*. That is, of course this can be done if we are willing to ignore the practice of mathematics, the epistemology of mathematics, obscure some of the content and *insist* on carving up the body of mathematical knowledge in terms of language, axioms and syntax. The advantage is precision in *comparing* theories. The disadvantage is disregard for the practice of mathematics, and loss of some content in the form of procedures and methodology.

The rest of the ways of individuating mathematics each have examples in the practice of mathematics, in the sense that one finds them in the opening pages of textbooks, in more casual presentations of mathematical theories, and in how it is that mathematicians think about the theory they are studying or working in. Practicing mathematicians sometimes individuate mathematical theories in terms of

(b)   the objects being studied,

such as when we start to study group theory or ring theory. It is not uncommon in such cases to be shown an example of a group, abstract away from it, and be told that we are studying objects of this sort. Practicing mathematicians also individuate mathematical theories in terms of

(c)   the operations,

such as when we study calculus, matrix algebra, the lambda calculus or functions; or in terms of

(d)   the context that sets parameters on what we may or may not do,

for example, when we move from Euclidean geometry to geodesic geometry, we may no longer extend lines indefinitely.

We might even individuate the 'same' theory in several ways. So, we might think of geometry *as* the study of the relationships between points, lines and plains, so, in terms of these *objects* of study. We then use these different geometries to study these objects under different circumstances, or different metaphysical or meta-mathematical assumptions about symmetry, the nature of space, straightness, orthogonality, parallelism and so on.

How mathematical theories are individuated is partly a matter of institutional conventions, history and content. Regardless, inconsistencies can arise within a theory or between theories. The inconsistencies can often be dissolved by adding precision or disambiguating, but not always.

2. Inconsistency Within a Mathematical Theory

For the most part, mathematicians are inconsistency *in*tolerant when it comes to inconsistency within a theory. When they find such an inconsistency they will usually:

(I)   abandon the theory, or

(II)   revise the theory to ensure against inconsistency.

Less usually, they might

(III)   adopt a paraconsistent underlying logic.

The last option is rarely chosen, but all three belong to coherent mathematical practice. Because of the last option, we do not want to say that mathematicians are wholly intolerant. How do they recognise an inconsistency?

In mathematics, an inconsistency is the same as a contradiction. Within a theory, these are formally identified. Constructivists define contradiction as a primitive entity, symbolised by 'bottom' i.e., $\bot$. In other formal systems it takes the form of a definition: *viz.* a formula is inconsistent or 'contradictory' iff it yields the equation '0 = 1'. In some logical systems it is defined as concerning objects: '$a \neq a$', and in others as concerning propositions: '$\sim P \& P$'. Whether we

spot these is sensitive to the precision of the language and definitions within a theory.

There are famous examples of contradictions in mathematics. Many appeared in the early developments of set theory. There are probably a very many contradictions that are not famous, since they never made it to print, because of mathematician's attitude of intolerance towards contradictions. We turn now to this attitude.

Consider the set-theoretic paradoxes. These arose at the conceptual level by asking questions about a mathematical concept such as: "Does the set of all infinite sets have itself as a member?" If it does, then it is never a completed 'set', since it keeps having to add itself in, as it were. If it does not, then it is missing at least one set, contradicting the universal quantifier used in defining the concept. The paradox only arises under the following assumption. Sets are fixed entities, that is, their members do not change over time or with the asking of a question! The paradox then becomes *alarming* under intolerance of inconsistency.

Alarmed, the community of classical mathematicians made more precise definitions of how to 'construct' sets, and hoped to *avoid* paradox and inconsistency. This is the classical reaction to paradox, and corresponds to (I) and (II) above. The set theories are elaborate and complex, and Gödel showed us that above a certain threshold of complexity, we cannot demonstrate the consistency of the theory within the theory itself. The best classical mathematicians could do, eventually, to be (relatively) certain of consistency was to make relative-consistency proofs: showing that if one formal representation of set theory is consistent then so is relative to another. This is far from the 'certainty' of proof of consistency within a theory.

Constructivists, of the *philosophical* stripe, as opposed to set theorists who 'construct' sets, and so also call themselves 'constructivists', have a different approach. The philosophical constructivists guarantee consistency by beginning only with formulas that they *know*, that are 'immediate', that is, they are so primitive that there is nothing more to say to justify them. They then derive, through *knowledge preserving rules*, further known formulas. The derived formulas are 'mediate' in the sense that they take mediation (manipulation through a rule, to get to them). Thus, they are assured of avoiding inconsistency from the start, so they have no need of equi-consistency proofs. They share with the classical mathematicians the intolerance towards inconsistency.

The third, paraconsistent response is to change the logic: the logical rules of inference or deduction at the object-level (within the theory), such that, *ex contradictione quodlibet* is not valid. This strategy amounts to the recognition that we can reason rigorously *through a* paradox. This is a type of theory revision as in (II), but it takes place at a deeper level, at that of the underlying logic. In the recent developments of such theories Mortensen (1995, 2010) and Weber (2013) make the logic explicit. But this last strategy might not be explicit in the presentation of a mathematical theory, especially if the theory is *not* identified by language, axioms and syntax. That is why the presentation of a theory, discussed at the beginning of the paper, is important for this discussion. Indeed, once we put on a paraconsistent hat, we start to see paraconsistent strategies, or interpret certain moves as paraconsistent even if they were not originally intended to be so (for reasons of anachronism, if nothing else), in several theories of mathematics, especially when they are presented in terms of objects, operations or context. That is, if we were to, say, rationally reconstruct collective mathematician's reasoning, then we might do so paraconsistently.[7]

For example, we might be interested in the theory of triangles, thought of as objects. Careful here, because it is difficult for those of us trained to think in terms of the different geometries presented by axioms and syntax, to think of the collective mathematical information we have about triangles as *a* theory, but try. In our general mathematical 'theory' of triangles, we learn that triangles sometimes have the interior angles adding to 180° sometimes less and sometimes more. It depends on the curvature of the surface. See figure 1. We also learn that they form a closed space, with a finite area, and that we can also think of them as open (as the area of the three 'outside' angles of the lines that form the closed triangle), and that these conceptions are equivalent for most of our equations. See figure 2. For example the interior/ exterior angles of the triangle still add up to 180° on a Euclidean surface. The same triangles will be judged 'similar', or form an equivalence class. Everything remains consistent up to measurement of area (since now the area of the 'triangle' of 'exterior' angles

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[7] There are such reconstructions in terms of the chunk and permeate strategy, but whether or not this counts as paraconsistent or rather as inconsistency avoidance is not decided in advance. Certainly, it can be blatantly paraconsistent, since within a chunk it is possible to adopt a paraconsistent logic, and we might even allow the information contained in the inconsistency to permeate from one chunk to another. However, if we deploy chunk and permeate to rationally reconstruct in a way that is loyal to the intentions and attitudes of the original mathematicians, then, in the great majority of the cases we will prefer classical reasoning within chunks. Whether we are reasoning paraconsistently at the meta-level when we reconstruct proofs within a *prima facie* inconsistent mathematical theory, is unclear.

will be infinite). We can think of the conflict between the 'area' of a triangle being both finite and infinite as a contradiction.
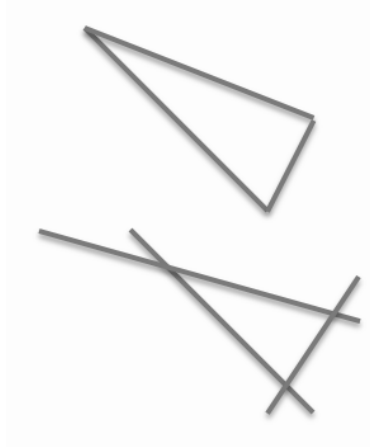

fig. 1

fig. 2

We can think of this theory of triangles classically or paraconsistently. If we think of it classically, then we go back to a more 'precise' explicit identification of theories in terms of language, axioms and syntax. And so we say that we have at least *two* theories of 'triangles' and we should not mix them, on pain of contradiction. We remove the threat of contradiction by compartmentalising at the level of individuating theories. Sometimes this strategy is adopted in chemistry, but rarely, since the theories are not so neatly individuated.

In contrast, if we retain the notion of a theory of triangles, and are then willing to think of the theory of triangles as holding a contradiction, then we can

take, say, an adaptive logic strategy:[8] accept that 'the theory of triangles' is fine most of the time, indeed it is fine up to measurement of area. We decide that inconsistencies, such as the one we encountered are examples of what we think of as 'abnormalities' and take recourse to the meta-level theory, which, say for the sake of argument, is classical. We then use the adaptive strategy of providing the context explicitly: whether we are thinking of triangles in the sense of closed finite area figures or as open, infinite area figures.

The two ways of thinking of our theory of triangles is (meta-)equivalent. In both cases we resolved the apparent contradiction in our theory of triangles. In the first we decided to disambiguate the notion of 'area' of a triangle by demarcating two theories, in the second we supply further information within the same theory, in this case, *the* theory of triangles. The contradiction remains in the theory, but *ex contradictione quodlibet* is invalid, and so we do not have a trivial theory of triangles.[9]

There are several ways of taking the paraconsistent route, each corresponding to a different paraconsistent logic, and each motivated by different philosophical, practical, historical, institutional or aesthetic considerations. What they together show is that it is (paraconsistently) possible to formally represent rigorous reasoning under circumstances of formally represented inconsistency *viz.* contradiction. Since we do sometimes reason in a way that seems to be rigorous (in the sense of admitting of recognisable correction) in the face of inconsistency, not just by putting the inconsistency in scare quotes, so naming it and not using it, but rather, reason directly *from* inconsistency in a perfectly understandable manner (not understood by those who have had a thorough training in classical logic, by the way) it makes (paraconsistent) sense that we should be able to represent that reasoning, since representing rigorous reasoning is what logicians have learned to do well.

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[8] An adaptive logic is one where there are at least two levels of theory. Each level may have a different underlying logic. At the 'working' or 'object' level, we carry on according to our logic, but 'outside' the logic, we identify 'abnormalities'. These are undesirable things that we may encounter in our work within the object-level logic. When we encounter these we adapt by going to the meta-level – where we have instructions about what to do. Typically, and in the case that we find interesting here, we identify 'abnormalities' with contradictions. We are then at liberty to leave the contradiction 'in' the theory (individuated as the mathematical theory of triangles), but interpret it adaptively, so there are two levels of theory.

[9] One of the things the paraconsistent logicians have taught us is to distinguish between a theory that has a contradiction and a trivial theory. The latter is one where every formula in the language of the theory is derivable and therefore true, along with, of course, its negation. Mathematicians are so far unwilling to knowingly work in a trivial theory. We might say that they, along with the great majority of people are intolerant towards trivial theories.

These were the more coherent and practiced responses of mathematicians when faced with inconsistencies within a theory. What of inconsistencies between theories?

2.2 Inconsistencies Between Theories

Inconsistency between theories is *not* the same as 'contradiction within a theory'. The difference is that these are much less problematic phenomenologically. They give less pause for thought because in the practice of mathematics, we are more adept at avoiding or resolving the inconsistencies between theories. Correspondingly, mathematicians are fairly tolerant towards inconsistencies between theories, but not completely, as we shall see, for, they are at great pains to keep the inconsistency *between* theories and not have it seep to *within* a theory. The pain is testimony to their only partial tolerance.
Inconsistency between theories is dealt with

(i)   by monism: abandoning one of the (each separately consistent or paraconsistent) theories, usually on grounds of intuition about the ultimate and single truth of the matter in mathematics, or in terms of what we can know in mathematics,

(ii)   by the careful cordoning off of formal theories from one another and then moving to the meta-level to compare theories, this is a type of moderate pluralism, or

(iii)   by a more radical pluralism which adopts a paraconsistent device or underlying logic at the meta-level of the theories in question.

The choice between these options is informed by deep metaphysical intuitions such as: essentialism, pragmatism, scepticism, anti-realism or quietism. Again, the paraconsistent strategy is rarely explicitly or intentionally adopted, but once if one, again, puts on one's paraconsistent hat, then one can think of that what mathematicians as are doing in such cases is adopting a type of paraconsistent strategy. But again, this would be disingenuous and not represent what mathematicians describe in their practice. Paraconsistent logic and reasoning is not yet popular or widespread.

How do we identify an inconsistency between theories in mathematics? The

answer will have to be 'in terms of contradictions'. We can be precise about this.

Theories are known to be inconsistent with each other just in case a pair of theorems, equations or axioms of the theories are individually known, are written in the same language and one can derive, or one has derived, a contradiction from the pair.

We wrote in the first section that there are different ways of individuating theories in mathematics. If we individuate theories by language, axioms and syntax, then identification of contradictions is straightforward provided we have a translation from the language of one theory to the language of the other. If the language of the theory is partly informal, such as we find in pre-twentieth century mathematical treatise, then 'the same language' qualifier should be understood as including natural languages. In this case, an example is that 8 is divisible by 3 in some theories but not others: namely, in those that lack fractions. This might not be apartment as a difference *in language* in an informal presentation of the operation of dividing numbers. If mathematical theories are individuated by the objects being studied, then inconsistency will consist in saying contradictory things about *an* object *supposed to be the same object* in both theories. We had an example in the last sub-section. Such inconsistencies are not a problem for the working mathematician. These are standard meta-theoretical statements, and they are plainly true or false, relative to the theory, and not both. Similar remarks apply to the other ways of individuating theories.

The practice of mathematics is not disturbed by contradictions between theories since mathematicians are pretty good at knowing in what context, or theory, they happen to be working. Problems can arise when they simultaneously draw on results or resources from different theories. They might do this in the course of a proof, explanation of a concept, in an application to a phenomenon in, say, science or engineering, and so on. This is *not* problematic when there is some affinity between the theories, and the affinity accounts for why drawing on the particular results or resources of different theories makes sense. In these unproblematic cases, it makes no sense to postulate a whole new theory on this particular occasion. We can even be quite tricky and use results or resources in a particular proof that deliberately and formally contradict what is said in another part of the proof, and do this without reasoning through the contradiction. In modern mathematics we see a lot of such borrowing and cross-referring. For a more thorough discussion see (Friend, 2017).

If we are concerned about this, we can show that this is legitimate by using

an adaptive logic, by using the strategy of chunk and permeate[10] to ensure the soundness of the overall proof or a mixture of both straegies. Lastly we might be using a paraconsistent logic to combine mutually inconsistent, or formally contradictory concepts, entities, operations and so on, or simply *avoid* using *ex contradictione quodlibet* to prove the theorem we want to prove. Because of the richness of the conceptual resources provided by paraconsistent logics, more-or-less widely known by mathematicians, then if they are worried, they can avail themselves of these resources and can be characteristically careful when mixing and matching from different theories. The reason they do this so little, is that it is only under the prompting of a problem arising that they will realise that they have not been carful enough (Sundholm, 2012). Otherwise, it is left to the philosopher, concerned with the practice of working mathematicians to reconstruct the reasoning using paraconsistent conceptions to *demonstrate* the coherence of inconsistency tolerance in the practice of mathematics or in the body of knowledge of mathematics.

3. Inconsistencies in Chemistry

In contrast to mathematics, 'theories' in chemistry are not so precisely individuated. They tend to be individuated in terms of types of reaction or types of object. For example, we might study acidity (a property of some chemicals), or we might study metals, or we might study processes, such as how certain types of chemical bonds are formed or dissolved. It follows that inconsistencies in chemistry are not always formally blatant or clear. We might have a theory of carbons or a theory of electro-chemistry, one is based on a particular set of materials/elements, the other on types of reactions and sub-atomic or sub-molecular theories. Inconsistencies might appear within a theory, between theories or between general metaphysical conceptions.

In the past, inconsistencies in chemistry have spurred research and have sometimes led to ideas being abandoned or in theory revision in a loose version of (I) and (II) above. It is loose because theories in chemistry are not always as formal as, in principle, mathematical theories are. Often what we find in

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[10] Chunk and permeate is a strategy developed by Brown and Priest, where proofs are divided into sections, called 'chunks' and bits of knowledge 'permeate' from one chunk to another. The bits that permeate are only those needed for the next chunk. Other bits are 'forgotten'. If we are clever about it, then contradictions never surface. Thus, the *order* in which one arranges the chunks and what information is in the chunk is very important.

chemistry is a pluralist attitude towards inconsistencies, or what is also called 'inconsistency tolerance', again, collectively, if not individually. We might, collectively, hold two mutually contradictory theories as hypotheses, and defer deciding between them until such time as it becomes important, or until such time as we can think of, or can afford to run, an experiment that will determine which of the two theories is supported by the observations in the new experiment. Of course in chemistry, theory determination rarely happens so simply, partly because of the vagueness in individuating theories and partly because the notion of 'observation' is partly informed *by* a theory.

I am new to the area of philosophy of chemistry, and so, cannot claim to have a thorough sense of the literature. However, what I have observed in the *philosophers* of chemistry I have read is that inconsistency is:

(1) deflected by calling on 'pragmatism', (Earley, Sr. 2015)

(2) deflected by calling on the notion of metaphor (Mahootian) or ambiguity (Byers 2011)

(3) by softening the appearance of inconsistency by being quite sophisticated and taking seriously the *language* in which we express the inconsistency; together with replacing the mode of 'fact' or 'absolute truth' in the relevant pronouncements in chemistry with more hypothetical or conditional modes in language (Chang), (Cellucci, 2017)

(4) softening the appearance by being careful about the 'representation' in language (formal or informal), but retaining a mathematically classical (inconsistency intolerant) reconstruction of the theories and their relationship to each other. For example, (Hettema 2015) uses the mathematical and conceptual apparatus of (classical) structuralism,

(5) by resorting to a paraconsistent strategy, such as that of chunk and permeate[11] to cordon off notions in chemistry in such a way as to always escape explicit inconsistency. (Martinez and Friend, in preparation).

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[11] Chunk and Permeate might be thought of as a paraconsistent strategy or not depending on what we think 'paraconsistency' means. The language has become slippery, of late. Here it is meant in the sense of 'as classical as possible', so with a fairly strong degree of intolerance towards contradiction.

While it might be hard to detect inconsistencies in chemistry, these strategies are all adopted in the spirit of 'rationality'. Moreover, there are strong metaphysical/ rational assumptions below the surface of these strategies. Where we locate inconsistency in chemistry will make a difference as to whether any of the above strategies is appropriate. The assumptions are disjunctive, that is, one might be made without the other or they might both be made. The assumptions are that the chemical world might simply be inconsistent. Or, that our representation of the chemical world through theory and language might be *inescapably* inconsistent. It is this bold proposal that I should like to entertain here, as a conceptual exercise.

3.1 Inconsistencies in Chemistry: Logical Tolerance of Inconsistency

What is the bold proposal?

Chemicals and their reactions might *themselves* be inconsistent.

Or, our *representation* of the chemical world through theory and language, be it formal or informal, *might be inescapably* inconsistent.

The disjunction looks a bit Kantian, and it is meant to. The first disjunct is very difficult for 'Western' people to cope with,[12] but let me just say this: *if* in virtue of the construction of our formal representation, we *foreclose* on the (paraconsistent ontological) possibility of inconsistent behaviour in the entities, substances or phenomena we study, then we foreclose *a priori* on the possibility of recognising such if it were to come up, be this in the future or retrospectively. This is simply a logical-conceptual point. Since we have developed the means to reason over inconsistencies, through the development of paraconsistent logic, there is no non-question begging logico-conceptual bar to our entertaining the thesis that the physical world itself, in particular the world of chemical substances, is simply sometimes inconsistent independent of us, our language and our theories. The more Kantian sounding version of this is that there is a possibility that the noumenal world is inconsistent. If it is, then if we foreclose on this possibility *a priori* then we limit our phenomenal understanding and risk error in our already limited understanding of the noumenal world.

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[12] An anonymous reviewer, could only understand the first disjunct *in terms of* the second. This is not what is meant, and it is not the intention. I really mean to propose that reality, the world we interact with is inconsistent. While, personally, I find this very difficult to make sense of, I would not then want to project my conceptual limitations on others. Just because I cannot make sense of something, it does not follow that no sense can be made of something.

Entertaining this first disjunct a little further: it is not the same as saying 'anything goes' in chemistry, or that there are no limits on possible behaviours. Such a conclusion drawn from the very idea of the first disjunct is a classical or constructive conclusion, and so begs the question. Moreover, the foreclosing of this possibility speaks against the supposed open-mindedness of scientists (which is counter-balanced by rigor of argument that is always wholly present when reasoning, paraconsistently, or not).

This is important because this foreclosing is exactly what many scientists have done in the case of, say, quantum mechanics, and the theory would take a new turn if we were conceptually prepared to be open to that possibility. I do not know if we are ready to do so or not.

Some quantum theorists are close to understanding this. For example, Abramsky is well aware of paradox and inconsistency in quantum mechanics. He draws a nice distinction in the reasoning between 'local consistency' and 'global inconsistency'. This is called the 'contextuality' of quantum mechanics. So, for him, the whole of quantum mechanics is inconsistent, (Abramsky et. al. 2015, 1) but pockets of reasoning are each locally consistent. [13] Moreover, he has a topological-logical meta-analysis of when it is that one approaching inconsistency. So the *recognition* of inconsistency is alive and well. However, his next reaction is interesting. He urges us to not *reason through* the inconsistency. That is, we reason short-of the inconsistency.

This is quite sensible in terms of methodology – of course, try to avoid inconsistency in the first instance since it is *difficult for us* to reason through inconsistency and hold on to the impression that we still *understand*. In other words, the phenomenology of such reasoning is unappealing to some people. That is not enough to rule it out in advance. In contrast, the reasoning is *not* unappealing to other people who have been trained in, say, paraconsistent reasoning directly and formally, or to people who have studied under certain schools of Buddhism [14] Personally, I think it is a matter of time before paraconsistent reasoning is accepted, or at least experimented with, in quantum mechanics.

[13] This is very close to the metaphorical version of 'chunk and permeate'. See Martinez and Friend, work in progress.
[14] For an example of the first, Priest explicitly declares that he is able to reason through inconsistency without losing a sense of meaning or understanding. In fact, he will claim to have better understanding for having carried out such reasoning. Mortensen would concur. For the second see (Engaging Buddhism. Jay Garfield.) His favourite school, and under his interpretation of the school: Dogen delight in paradox and use it to attain better, more profound 'understanding', all though they might (depending on who they are talking to, and what they think is the better strategy for helping their interlocutor attain better understanding) qualify this by saying that the very notion of understanding has shifted. And it might well have.

This should not be too alarming. After all, it is not as though the contradictory phenomena in quantum mechanics are completely chaotic and unpredictable. It is not the case that we have a trivial (or logically explosive) particle or a wave that is also an elephant and a poem. The possibilities of behaviour of entities in quantum theory are very limited. There are parameters around their behaviour. It is just that we do not know in a particular instance whether the phenomenon will be like this or like that. Moreover, once we open up our minds to the possibility that they might be both at the same time, then we might find that grosser *chemical* reactions between substances, compounds or relatively pure samples of elements are also inconsistent or are reasoned over *better* (where 'better' is a matter of taste, training, inconsistency tolerance and so on) with a logic and language that *allows* inconsistency but not explosion. We would then be able to observe inconsistencies in reactions in chemistry, for example, and by this, I mean that we would observe the inconsistency in itself, as a real phenomenon, as part of reality. The 'observation', of course, is hostage to our language and logic and inconsistency tolerance, but once we allow for inconsistency in our language, we can then observe inconsistency.

It also works the other way around. If we are inconsistency tolerant, then we might put pressure on the language, logic and theory to accommodate the observation. Whether we start with the language and theory or with the observation is neither here nor there. The two sit in close proximity. Nevertheless, we should not foreclose on the *prima facie* possibility of 'observing' an inconsistency in reality, whatever that is like. Our training and conditioning might *now* prevent us from making or even seriously entertaining such observations, but that is not enough to rule out the possibility that they could be had, under different conditioning. 'Observation' is not a term innocent of theory, language and metaphysical or logical scruples. Let us end our consideration of the first disjunct of the proposal with this statement: in a paraconsistent sense of 'possible' it is possible for objects, substances, waves, relations, systems or phenomena to be inconsistent. Therefore, as good scientists, we should not foreclose on this possibility by constraining our language or inference rules. Enough said about reality being inconsistent.

Now for the second disjunct. What does it amount to? We represent chemicals and their reactions using theories. The theories are metaphysically

informed and written in a language,[15] or several languages, often a mixture of formal and natural languages. Since we are discussing at the level of language now, we keep our definition of 'inconsistency' from mathematics: namely it is the same as 'contradiction'. Contradictions are expressed and recognised *in a language*. We rarely have strict identity in chemistry (Needham, 2015). Rather, we have equivalence up to: our purposes, sensitivities, measurement and relevance of context. So any contradiction might be in the language and theory but not in but not in reality as such. Does it follow that the representation (theory and language) *should* accommodate inconsistencies, thus making provision for the inescapability of inconsistency?

It might not be a bad idea *ab initio* to try to fix the representation and keep it consistent: check for vagueness, ambiguity, contextual differences etc. but it might not be a bad idea *later*, to not worry too much about the inconsistencies. In fact, this is what we observe in the practice: different degrees of tolerance towards inconsistencies. As good open-minded scientists, we cannot decide *in advance* that our reasoning has to be consistent, in contradistinction to rigorous! Here, 'rigorous' just means prone to recognisable error and subject to agreed correction. For example, we regularly reason over margins of unpredictability when we reason about stochastic processes, and such reasoning can be inconsistent. One model might tell us that there is a 60% chance of rain today, and another, equally good model tells us that there is a 30% chance of rain. Which is it? It cannot be both, and no particular observation of the weather today can tell us which. There are similar examples in chemistry: one model tells us that there is a 60% chance of reaction x occurring and another model that there is only a 30% chance. The experiment might be difficult or costly to run, and so we do not have enough data to decide between the models. Or, it might not be a question of sufficient data at all. After all, we have not decided on one model for the weather, and we have plenty of data. When we meet such data, we might apologise and say that stochastic talk is a patch – the best we can do, since the

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[15] We might also note that language is metaphysically informed, and that metaphysics is linguistically informed. For an example of the first: we do not bother to have one word for the top half of a medium-sized dry object. The top half of such objects is metaphysically not very interesting, and we rarely need to refer to them, so to award them a particular word would be too generous. For an example of the second: we are sometimes seduced by grammar – to refer to Wittgenstein. Or, rather we are sometimes seduced also by the fact that in the language we are most familiar with there is one word for something, and we naïvely expect for there to be one corresponding sort of object. Because we have the word 'chemical' we think that there is a type of thing at some very general level that we call a chemical.

world is (*a priori*?) deterministic, causality linear and inconsistent phenomena are impossible. In other words, we might attribute inconsistency *to* our representations without compromising our metaphysical convictions about the consistency of 'reality'. The inconsistent representation is hostage to the (inevitable or temporary)[16] gap between the metaphysically consistent reality and our inconsistent representation. And there are many places to look to improve the representation: vagueness (eg. impurity of substances) or to the *ceteris paribus* clauses that sit behind our representations in language (Llored, 2015). But, we cannot rule out *in advance* that inconsistent representation is intolerable. In fact, some philosophers encourage it. For, it is exactly the inconsistencies that lead to refinements in our theories. (Byers 2011) So the inconsistency is fruitful in exactly this respect.

Let us consider an example.[17] We might start our study of chemistry with a metaphysically, or mereologically, naïve conception of chemicals as being made up of parts, say, molecules, and these parts, are made up of other parts – atoms. Atoms are made up of neurons, protons and electrons. They differ from each other in the ratio of these parts to each other. We then naïvely expect that combining chemicals is analogous to combining a bunch of stones at room temperature. They just sit there together and only interact with each other in terms of relative position, one might sit on top of two others. We have a heap of stones, and they are stable unless we move the base upon which they rest. We give up on this conception very quickly in chemistry. Liquids and gasses do not only combine and form a heap. They take time to separate out, if they do so at all, and they interact with each other. The whole might be less, or more than the sum of the parts, in volume, mass, heat and so on. Furthermore, the order or rate of combination, in some cases, will also affect the sum of the whole.

'Affordances' is a word used to explain this. Affordances are potentials or predispositions that are sensitive to, or vary with, both "the way it is interacted with and ... the context in which the interaction takes place." (Harré, 2015, 107). We might mix liquids quickly or slowly (the way they are interacted with) or we might mix them in a 'context' of very low pressure, high temperature and so on. Past certain parameters, the resulting mixture will have very different properties.

Let us now reconstruct the example in terms of our second disjunct. We are

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[16] Complexity theory would have us think that the presence of the gap is inevitable. Language is finite and limited, and therefore, cannot for simple mathematical reasons, tell us all about a continuous reality. (Needham, 2015)

[17] An anonymous reviewer suggested the example. I am grateful for the suggestion.

faced with a contradiction between our naïve metaphysical and mereological view on the one hand and our experience and observations and measurements on the other. When we try to make mayonnaise, we cannot just add the oil to the egg yolk. We have to stir and we have to stir within some margins of rate. We also have to add the oil slowly, again within limits – an upper and lower bound. We can make mayonnaise or we can make a rather unpleasant separated-mixture of egg yolk and oil (an unsuccessful attempt at making mayonnaise, call this a 'uamm'). The two: mayonnaise and uamm, have different properties, textures, volume and flavour. Methodologically, we adjust our naïve conception, we introduce a new term to the language: 'affordances'. Egg yolk has the affordance of combining with oil to make mayonnaise. It also has the affordance of combining with oil to make uamm. Contradiction has been abated. This is good methodology. Try to get rid of the contradiction. However, note that the contradiction was there for a period of time. The introduction of 'affordances' to the language did not happen the first time we made mayonnaise and realised a discrepancy between what we expected under our naïve view and our experiments with trying to make mayonnaise, when we sometimes made uamm instead. We develop our knowledge of chemistry through experiment. Moreover, we cannot always anticipate the affordances of a chemical based on its composition. There might be affordances we never detect. There might be affordances that are such that we cannot work out what the relevant 'way of interacting' is or what the relevant context is that has a chemical or mixture react in one way or another. This might be a matter of time, of developing our science further. We analyse one phenomenon at a time, according to our competing interests and within practical limitations. But the point is that there is more than this. The limitations that can be, and should be, gradually pushed back are also theoretical and linguistic, and before they are pushed back, there is a contradiction with which we struggle.

Moreover, practicing inconsistency tolerance earlier, might lead to more fruitful discoveries in our science. Formal representation and even representation in a natural language hides aspects of 'reality'.

Descriptions are always incomplete. On the one hand, this is their strength. We cut out what is irrelevant to our reasoning or interests. But when the representations are inconsistent, and there is no logical explosion, this is a nice opportunity to become more explicit and to re-examine the *ceteris paribus* clause in our *represented* inconsistency. We might even be so bold as to acknowledge this at the logical level in our reasoning (in a language) in chemistry,

and take a more radical route than those mentioned above. If we really want to practice inconsistency tolerance, be it temporarily, or in the hopes that it will later go away, but we recognise that it is useful or fruitful to keep the inconsistency around; then we should take a more obviously paraconsistent pluralist route to understanding in chemistry. We might adopt a paraconsistent logic that explicitly forbids *ex contradiction quodlibet* inferences and is rigorous. Who knows, we might even go so far as to change our reticence and accept that reality itself might be inconsistent.

Conclusion

Having discussed inconsistency in mathematics and inconsistency in chemistry separately, I now ask the question whether one set of reactions by mathematicians or chemists can inform the other's practice and theory development. I draw on Friend (2014) and Byers (2011) for the account of inconsistency and pluralism in mathematics, and on Chang (2012), Schummer (2015) and Byers (2011) for the account of inconsistency and pluralism in chemistry.

The mathematician's strategies are rarely adopted by chemists, but are each available. The first two are: theory abandonment and theory revision. The drawback for the chemist in deploying these strategies is that they depend on being able to individuate theories in chemistry in a very precise way. It is difficult to do this, and stay loyal to the practice. Arguably, the flexibility in theory boundary and in representation are *necessary* for the advancement of chemistry. Nevertheless, the advancement is often accompanied by increased precision in formal representation, definitions, symbols and so on. I do not advise the chemist to adopt either of these strategies *without* simultaneously paying attention to the strategies of deflection and softening. But maybe the chemist would be well served to adopt the third: the pluralist paraconsistent strategy. This would give a flexible but transparent rigour to the reasoning in chemistry.

The other way around: from how inconsistencies are handled in chemistry to how they could be handled in mathematics is more interesting. First, in the practice of chemistry (as opposed to the philosophy of chemistry), there is an *ignoring* of the *classical* logical consequences of inconsistency, although there is a phenomenological un-ease. This suggests to the mathematicians that they might relax their *doxastic* fixation on consistency. This ignoring without further ado is not available to the mathematicians. Instead, second, inconsistencies in

chemistry are the source of disagreement in the community, and also the source of innovation. Moreover, as Chang argues, keeping conflicting metaphysical conceptions around, and alive for longer – what he calls a 'pluralism' in chemistry, leads to more fruitfulness in chemistry, not less. Thus, on the part of Chang, Schummer and Byers there is a deliberate recognition of the important *role* of inconsistency in the science of chemistry. Byers argues that this is also the case in mathematics, but reports that mathematicians are reluctant to acknowledge this. Maybe mathematicians could learn from the example of chemistry, but take the issue further and be more precise, and return to the chemists with their fine conceptual tools.

Interestingly, when there is disagreement in chemistry, especially in the form of an inconsistency, this takes a political and institutional turn. But for the discipline as a whole, it is better to keep the inconsistency alive. Maintain the tension, and encourage new concepts. Thus, with inconsistency we have local and individual discomfort, but from the point of view of the discipline, a flourishing. This point is eloquently made by Byers in the case of both mathematics and science.

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