Ontology and Mathematics in Classical Field Theories and in Quantum Mechanics

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ABSTRACT

A draft of a possible comparison between the use made of mathematics in classical field theories and in quantum mechanics is presented. Hilbert's space formalism, although not only elegant and powerful but intuitive as well, does not give us a spatio-temporal representation of physical events. The picture of the electromagnetic field as an entity which is real in itself – i.e., as a wave without support – fostered by the emergence of special relativity can be seen as the first step, favored by many physicists and philosophers, of a gradual "escape" from intuition into a purely mathematical representation of the external world. After the introduction, in recent theoretical physics, of fiber bundle formalism the classical notion of field acquires a new spatio-temporal intuitiveness. This intuitiveness is clearly foreshadowed in the Kantian and Meinongian analysis of the notion of magnitude. At the end of the paper we show that, contrary to what happens in quantum mechanics, mathematics plays a truly explicative role in general relativity, without any loss of spatio-temporal intuitiveness.

INTRODUCTION

In the chapter on "Field and ether" of their popular book, Einstein and Infeld (1938) write that to the modern physicist the electromagnetic field is as real as the chair on which he sits. It is easy to find statements of this kind in handbooks

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of physics, but we mentioned Einstein because he was the first to show the truth of this statement, by introducing the special relativity theory. As a matter of fact, at least until 1905 but even after that, many physicists considered electromagnetic waves to be similar to sea waves and sound waves, that is they thought of them as oscillations of some kind of matter. Nevertheless it was not so easy to identify the carrier of electromagnetic waves, whereas for sea and sound waves they are, respectively, water and air; due to the enigmatic character of the stuff which transports light, it was dubbed "ether".[1]

The vanishing of luminipherous ether from electromagnetic theory brings with it the idea that waves are realities in themselves without support. But it is easy to understand what an ether wave is, on the basis of the analogy with a sea wave, whereas it is quite difficult to visualize a wave that isn't a wave of anything. Therefore one can state that with the coming of special relativity theory and the refutation of the hypothesis that electromagnetic waves are the oscillation of some sort of matter, a "withdrawal"[2] was begun in the direction of a mere mathematical representation of what happens in the physical world. This withdrawal was favored by many physicists and philosophers and reached its apex in the renunciation of the possibility of a spatio-temporal picture of what occurs in the microphysical world after the advent of quantum mechanics.

It is clear that mathematics is essential to physics, but no type of formalism is suitable for every domain of objects. We will show that only with the introduction of fiber bundles does Einstein's above mentioned statement become fully understandable. Nevertheless not every implementation of new mathematics favors the comprehension of physical reality. For instance we will see that in quantum mechanics the formalism, though elegant and powerful, in a certain sense, overshadows, instead of shedding light on, physical reality.

In the first section below the sense in which Hilbert's space formalism intuitiveness is not really physical is presented. In the following, on the contrary, it appears that fiber bundles clarify the notion of the classical wave. In the third section the importance of the notion of intensive magnitudes for this clarification emerges. Finally we present the intuitive character of general relativity moving from the concept of tensor in elasticity theory.

[1] On these topics see the still wonderful Whittaker 1951.

[2] See for instance Heisenberg 1958, ch. 10. The history of the "withdrawal" is told very well by Hanson 1963, albeit in a little too hagiographic way for our taste.

QUANTUM MECHANICS

If on entering a café we ask for the rest room and we are told to go first left and then right, but we go first right and then left instead, we usually end up in a different place. This example shows that though sum and multiplication are commutative operations, sometimes by inverting the order of an operation we change the result. Here the operation is the composition of two translations on the plane. This is also true in the case of rotations in space. Indeed in algebra the rotation of a vector (*x,y,z*) around the *z* axis is represented by a matrix of the type:

$$R_\theta = \begin{vmatrix} \cos\theta & \text{sen}\,\theta & 0 \\ -\text{sen}\,\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix},$$

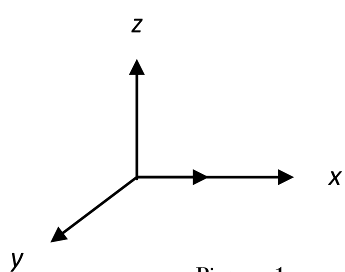
where "ϑ" is the rotation angle. If R_φ is the matrix representing the rotation of angle "φ" around the *x* axis, then:

$$R_\varphi = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \text{sen}\,\varphi \\ 0 & -\text{sen}\,\varphi & \cos\varphi \end{vmatrix}$$

Generally it holds:

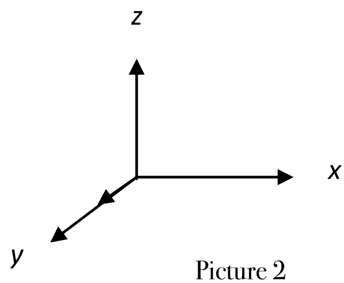
$$R_\theta R_\phi \neq R_\phi R_\theta\,,$$

which means that the composition of two rotations in space – one around the *z* axis and the other around the *x* axis – is a non commutative operation. Let us propose an example. Let us consider a vector lying on the *x* axis:



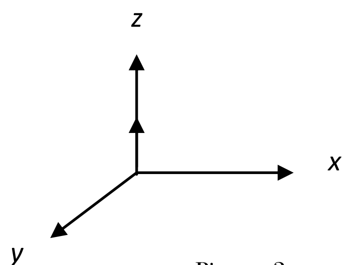
Picture 1

If we rotate it clockwise by a right angle we obtain:



Picture 2

That is, the vector now lies on the *y* axis. Then we rotate the vector by a right angle clockwise around the *x* axis and we obtain:



Picture 3

Returning now to the situation of picture 1, we invert the order of rotations; that is, first we rotate the vector by a right angle clockwise around the *x* axis. Obviously, since the vector lies on the *x* axis itself, such a rotation does not produce any effect. Therefore we stay still in the situation of picture 1. Then we rotate the vector by a right angle clockwise around the *z* axis and we obtain the situation presented in picture 2, that is the vector lies on the *y* axis. It follows that, since in the first case the final configuration is that of picture 3, by inverting the order of rotations the result changes.[3]

When Heisenberg realized that the *p* and *q* variables, i.e., moment and position, do not commute in the new quantum mechanics, Max Born proposed representing the new kind of entity no longer through a number, as in classical

[3] Remember that, on the contrary, rotations on the plane commute.

physics, but through a matrix.[4] Thus in new physics a new kind of mathematics was introduced, which had been developed many decades before, in order to deal with rotations in space. At first the new formalism seems very abstract. Indeed Heisenberg repeated many times that in the mathematical representation of nature we have to abandon spatio-temporal intuition. Nevertheless the intuitiveness of the new formalism appears again when von Neumann interpreted Schrödinger's equation – that is, the law which governs the evolution of quantum state – as a rotation in Hilbert's space (von Neumann 1932, §.III.1). Many handbooks of quantum mechanics emphasize, in fact, the possibility of representing geometrically what occurs through Hilbert's space picture (e.g., Fano 1971). Yet we have to remember that the space in which such processes happen is not a physical one, but a mathematical manifold without representative capacity.

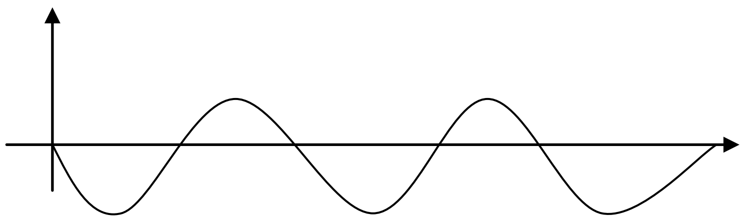
In a certain sense one can now formulate the great unsolved problem of microphysics – i.e., the measurement muddle – as the impossibility to come back from the abstract representation in Hilbert's space to what happens in physical space. For instance in a 2-dimensional Hilbert's space, a dichotomic variable, such as the spin of one particle, can have at the same moment two values, even though each one is not completely ascribed to the particle, but only partly, determined by its "probability amplitude". Nevertheless, when one measures spin in physical space, the particle shows only one of the two values; but to identify which presumptive physical process would allow us to abandon the representation given by the superposition of the two states, returning to the more prosaic determination of the physical reality, is not easy.[5] Indeed one forgets that, although mathematical language is absolutely essential to the physical understanding of the world, it is nevertheless dangerous to state that there is only a mathematical explanation of what happens, i.e., an explanation totally deprived of intuitive character. Such a perspective favors Platonist and instrumentalist interpretations of contemporary physics: i.e., either theoretical terms of physics represent entities completely different from those of our experience, or they have no representative capacity. Thus one opens the Pandora's box of the wildest speculations of contemporary physics, lamented, for instance, by Smolin (2006).

[4] See Cassidy 1992, ch. 10. See also Born (1978, ch. 19) where the German physicist tells how, probably on July 10th 1925, he noticed that the kind of multiplication, which Heisenberg needs, was that between matrices.

[5] See, for example, Ghirardi 1997.

THE ELECTROMAGNETIC FIELD

As mentioned above, this escape into mathematical representation begins when one affirms that the electromagnetic field is not a wave of any stuff, but is real in itself. On the other hand it is usual to represent an electromagnetic field as a wave in physical space through drawings of this kind[6]:



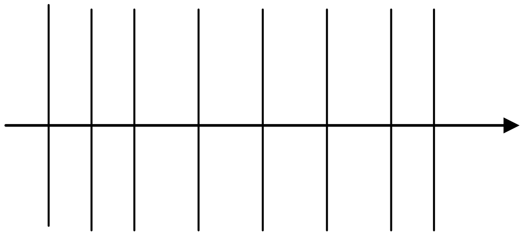
Picture 4

The apparent "crests" and "valleys" lie in physical space, as happens in the case of sea waves, but nothing oscillates in space! Therefore one is compelled to ask: waves made of what? The standard answer is that it is not possible to find an intuitive picture of what happens, because these drawings have only a heuristic and didactic value, but they are not able to express what actually occurs. Indeed, according to physicists, only the mathematics of Maxwell's differential equations provides a suitable representation of reality, that is a non-intuitive representation of it.

On the other hand, in our opinion, at least for classical field theories, the situation is quite different. In picture 4 something goes wrong, i.e., there are waves, but they are not extended into physical space. A different kind of mathematics can help provide a more intuitive representation. To understand the point let us consider a 1-dimensional physical space. Then let us ascribe to each point of this straight line a 1-dimensional vector space. Roughly speaking one can image the situation as a bundle of infinite parallel straight lines all perpendicular[7] to the unique spatial dimension we are considering.

[6] For the sake of simplicity we have represented the wave in 1-dimension.

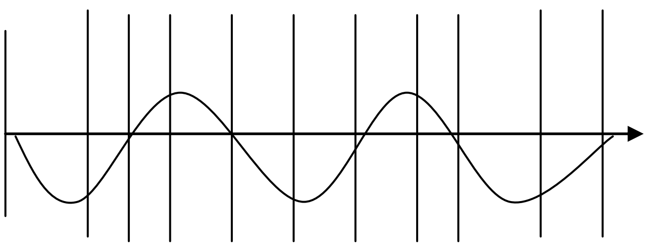
[7] This perpendicularity is not essential to all fiber bundles, but only when fibers are tangent spaces.



Picture 5

In other words, each point of the basis must correspond to a straight line. Now let us associate each point of the basis to a vector, which could represent, for instance, an electric field.

Each point of the 1-dimensional space has a determined value of the electric field, to which a given vector corresponds.[8] One can put the latter inside each "fiber" (as the infinite perpendicular straight lines are dubbed). Probably the electric field varies like a wave along the straight line. But we now understand where the wave lies:



Picture 6

That is, as is evident in picture 6, the electric field oscillates in the fiber bundle not in physical space.

It is easy to generalize such a reasoning to the case of a 3-dimensional physical space and therefore to 3-dimensional vectors, which lie in 3-

[8] In the 1-dimensional physical space the electric field is represented by a 1-dimensional vector.

dimensional fibers. In a fiber bundle it is evident how the electromagnetic field is real in itself, that is it doesn't need a physical support such as ether.[9]

One could say that this mathematical representation is even more artificial than that of Maxwell. Indeed one might ask where are these three additional dimensions, which one has to ascribe to each point of physical space? Yet all this is quite intuitive. In the *Critique of Pure Reason* (B202ff.) Kant already distinguished between intuitions (*Anschauungen*) and anticipations (*Antizipazionen*) of perception: the former are extensive quantities – extended in space and time – the latter are intensive quantities, which are ascribed to a single spatio-temporal point. To understand this, let us consider a yellow colored stripe, which passes slowly from left to right from a milder yellow to a darker one. In this situation color has an intensity peculiar to each point of the stripe, as in the case of electric field. In general it is quite reasonable that the complexity of "reality"[10], at each spatio-temporal point, is so articulated as to require a rich fiber space to represent it. The reality that, as it were, fills our spatio-temporal intuitions possesses different qualities, such as yellow and red, smooth and rough. Moreover these qualities present different degrees: more or less red, more or less smooth. Something similar occurs in the representation of an electromagnetic field through fiber bundle geometry.

INTENSIVE MAGNITUDES

As we were saying at the beginning of the present paper, in order to play a truly explanatory role and to ensure the cognitive power of physics, theories, in their attempts to describe reality, ought to make use of what we might call the "appropriate" mathematics. By "appropriate" mathematics we mean those mathematical theories which seem to be closer to our empirical intuitions. We have also seen that the topological notion of a fiber bundle provides a good example of "appropriate" mathematics. As a matter of fact, contrary to Euclidean space or to Minkowsky space-time, fiber bundles constitute an intuitive representation of electromagnetic waves and help us to grasp the sense in which it is said that electromagnetic waves are real.

[9] For an intuitive and elegant presentation of the role of fiber bundles in contemporary theoretical physics see Penrose 2004, ch. 15.

[10] *Realität*; it is Kant's terminology.

In this section we would like to suggest that the definition of *intensive* magnitude given by Kant might be held to provide the intuitive ground for the mathematical description of classical field theories in terms of fiber bundles. In order to do this, a few historical considerations might be helpful.

By the second half of the nineteenth century experimental psychology had opened a whole new chapter in the study of the mind by showing the possibility to introduce an experimental approach to the matter. This attempt soon raised a heated epistemological debate concerning the limits of measurement procedures and the definition of measurement itself. As a consequence of the reductionism inherent in any experimental approach, the *qualitative* side of sensations dropped out of the picture and the new psychology started considering sensations as a particular kind of *magnitude*. Positivist epistemology had also adopted Kant's distinction between "extensive" and "intensive" magnitudes together with the corresponding definitions provided by the first *Critique*. According to these definitions, sensations had to be regarded as intensive magnitudes: «A magnitude», Kant wrote «that is apprehended only as a unity, and in which multiplicity can be presented only by approaching [from the given magnitude] toward negation, = 0, I call an *intensive magnitude*» (Kant B 210). The possibility to measure something directly or indirectly allows a distinction between "fundamental" and "derivative" magnitudes. In fact, in the case of indirect measurement, what we are really measuring is not the magnitude we are interested in, but some other magnitudes, which have the empirical feature of being functionally related to it.

The prevailing stance within the epistemological debate mentioned above can be traced back to the work of Hermann von Helmholtz (1821-1894). According to Helmholtz (1887) extensive magnitudes only enjoy the status of "fundamental" magnitudes, whereas intensive magnitudes ought to be considered and treated as merely "derivative" ones. A good example of this attitude towards magnitudes can be found in the ideas concerning measurement endorsed by Helmholtz's pupil Johannes von Kries (1882). The intensive magnitudes that physics deals with, according to Kries, are merely "combined units" (*combinierte Einheiten*), i.e., they are made up of (and are nothing more than) the three fundamental physical dimensions of space, time and mass, and they are therefore considered to be the result of a general agreement regarding their practical applicability. From his point of view then, the notion of "intensive magnitude" constitutes a sort of derivative concept, a concept one cannot grasp without the help of the more fundamental and intuitively evident notion of "extensive magnitude".[11]

Towards the end of the nineteenth century, an unexpected turn was given to the debate on magnitudes by the Austrian psychologist and philosopher Alexius Meinong (1853-1920). Meinong had studied in Vienna as a pupil of Franz Brentano (1938-1917) and had founded in 1894 in Graz the first Austrian laboratory for experimental psychology. In 1896 he published a long essay with the title *On the Meaning of Weber's Law. Contributions to the Psychology of Comparing and Measuring*. In this essay, rejecting the view endorsed by Kries and generally accepted by physicists, Meinong suggested a radical revision of the debate concerning measurement. Measurement, he believed, must be considered primarily as a mental process and its logical analysis should therefore come only after a purely psychological account of its main phenomenological features. Moreover this logical analysis, in order to be grounded in our intuitions, has to account for those psychological features. Meinong's treatment of measurement commits him therefore to the view according to which intensive magnitudes must be regarded as fundamental. As a consequence, extensive magnitudes should be thought of and treated as derivative magnitudes.

The logical analysis of measurement, in other words, has to mirror specific features made visible by its psychological description and this includes accounting for the new relationship that now occurs between the two kinds of magnitudes. In practice, this amounts to acknowledging the fundamental status of intensive magnitudes and to giving a definition of the general notion of "magnitude" which encompasses in itself the case of extensive magnitudes as well. The definition given by Meinong is the following: «If one can think of a *y* such that, so to speak, if seen from *x* it lies in the same direction as *non-x*, then *x* has or is a magnitude and *non-x* is zero» (Meinong 1896, §.1).[12]

It might be helpful at this point to try to render explicit the tacit, but deeply intuitive, assumptions that underlie this apparently mysterious definition. According to Meinong, a thorough psychological examination of our

[11] Kries assumes here that space, time and mass are extensive magnitudes. We would like to point out, though, that while the inertial mass (*quantitas materiae*) is an extensive magnitude, the gravitational one may also not be so.

[12] It should nevertheless be kept in mind that Meinong didn't think of the above formulation as of a fully fledged "definition". He would regard it rather as an empirical criterion useful to distinguish between things that can be considered magnitudes from things that cannot.

perceptual field reveals the presence of certain "directions" within this field. The idea is that if we start by considering the *absence* of a particular phenomenon these directions allow us to order a series of phenomena similar to it. As everyday language shows, in the case of magnitudes the notion of "dissimilarity" (*Verschiedenheit*) itself entails the idea of a "direction". If, by comparing two magnitudes *A* and *B*, we found that they are not identical, we probably wouldn't say that "*A* and *B* are dissimilar", but rather that "*A* is bigger (or smaller) than *B*". The reason for this is that the second expression allows us to assign to *A* and *B* a specific position on the imaginary line which connects them to the zero point.

According to the general definition of magnitude suggested by Meinong then, there are, within our perceptual field, several imaginary non-overlapping "lines" along which different kinds of magnitudes (spatial, temporal, intensive magnitudes etc.) draw near to each other toward their zero point. To use Meinong's own words a magnitude is the "disposition" (*Eignung*) of a *quality* to belong to one of the many lines which converge toward a zero point. It is precisely in virtue of this capacity to move along this imaginary line which connects it to its zero point that we can attribute a magnitude to a given quality. The general condition that has to be satisfied in order for two magnitudes to be comparable is the fact of lying on the *same* line. This means for instance that two points a' and a'' which lie on different lines are certainly sufficient to determine a specific direction, but this direction doesn't approach any zero point and the two points cannot therefore be compared.[13]

The above considerations seem to us sufficient to support the view that the notion of magnitude does not necessarily have to be bound to any extensive representation of space or time. Magnitude can also be seen as a sort of degree which takes place in space and time and which is capable of varying from one point to the other. We consider fiber bundles as an "appropriate" mathematical instrument, useful to connect classical field theories to our own intuitions concerning the external world.

[13] See Meinong 1896, §.7. It is worth pointing out that Meinong's approach was partially adopted by Russell, in 1903, but since then has never been seriously considered in subsequent epistemological debate on this same topic and would therefore merit more careful analysis. But see Fano 1999.

GENERAL RELATIVITY

Quantum mechanics and general relativity theory are the most important theories of contemporary physics. As is well known, till now it has not been possible to conciliate them. Both make a massive use of mathematics, but in the former, as mentioned above, formalization partly hides an as yet unresolved difficulty, whereas in the latter, as we shall see, mathematics is a powerful instrument useful to specify our intuition.

An essential peculiarity of general relativity is the dependence of physical space metric on the distribution of masses. In classical physics the distance between two points of coordinates (x_1,y_1,z_1) and (x_2,y_2,z_2) is given by:

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2} \quad (1)$$

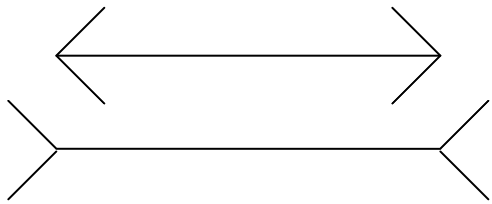
On the contrary, in general relativity, for each couple of points, distance is evaluated by a much more general formula:

$$\sqrt{g_{xx}\Delta x^2+g_{yy}\Delta y^2+g_{zz}\Delta z^2+g_{xy}\Delta x\Delta y+g_{xz}\Delta x\Delta z+g_{zy}\Delta z\Delta y} \quad (2)$$

Where, as usual, the Δs indicate differences between coordinates. In a certain sense (2) is a generalization of Pythagoras's theorem, which we apply in (1). In relativistic space[14] coefficients g_{ii} could always change point by point. For this reason space is no longer mathematically represented by an Euclidean space, but by a more general geometrical entity dubbed "differentiable manifold", for which the metric is not constant, so that it is possible to ascribe to each point the correct series of coefficients g_{ii}, that is the metric peculiar to that point. Such metric depends on Einstein's celebrated equation, which allows us to calculate[15] coefficients g_{ii} given the distribution of masses and energy[16].

Contrary to the first impression, representing space by a geometry, in which the metric varies point by point, is not against our intuition. For instance, in our perception, the same couple of points, if embedded in different pictures, could appear nearer or farther, as occurs in the famous Müller-Lyer illusion:

[14] Here for the sake of simplicity we do not consider the 4th dimension, that is the temporal one.
[15] Such a calculation is actually quite difficult and often does not produce univocal results.
[16] For an intuitive and more exhaustive presentation see Penrose 2004, ch. 19.



Picture 7

Therefore perceptual space does not have a constant metric, so that representing physical space with a varying metric is not so bizarre, because the same, in a certain sense, occurs for visual space. Of course the former variation of metric is much more objective.

We have seen that in classical theory of the electromagnetic field it is possible to ascribe to each point of physical space a 3-dimensional vector space, in which, as it were, the vector electromagnetic field lives. Nevertheless we already know from elasticity theory that each point of physical space could possess complex properties, which could not be represented by a simple 3-dimensional vector. For instance let us consider a rock salt crystal; if one applies a shear force parallel to the crystal plane, the stone quite easily cleaves, whereas, if *at the same point* one applies a force perpendicular to the crystal plane, the stone opposes more resistance and in the end it breaks. Therefore the resistance to stress of the crystal at that point is not isotropic, so that to represent in physical theory a much more complex situation, a vector is no longer sufficient. For this reason already at the end of nineteenth century physicists[17] introduced the concept of *tensor*, which is able to ascribe to a point of physical space as many variables as one wills.[18]

All these values could, as it were, live in multi-dimensional fibers, which one applies to each point of physical space, as we have done in the case of the electromagnetic field, but in a more general form. Neither is this against intuition, if one considers that we are, for instance, able to distinguish hundreds of thousands of different hues of color, which could be psycho-physically ordered on the basis of many variables. And this holds for color, but our perceptual space is much more complex. Therefore the qualitative

[17] See Voigt 1898.
[18] On this topic Brillouin 1938 is still valuable.

character of perceptual space has a very articulated structure, so that it is not so strange that in physical theory one is compelled to introduce such complex fibers to represent what occurs.

All this is of utmost importance for general relativity, because Einstein's equations connect the metric tensor, i.e., the set of g_{ij} coefficients – which changes point by point – with the energy-moment tensor, which describes the distribution of masses and energies. It is thus evident that the mathematization implicit in the theory, though not simple, stems from our intuition; therefore formalization here facilitates our understanding, without overshadowing unsolved problems. As we have seen, the same does not hold for quantum mechanics. Our opinion with respect to the latter theory is similar to that expressed at the beginning of the paper for Einstein's statement that the electromagnetic field is real in itself, before we found the more intuitive representation of fiber bundles. It follows that, as far as the microphysical world is concerned, either we have not yet identified a suitable formalism, or there are some physical phenomena which still escape our understanding.

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