

An Abstract Mereology for Meinongian Objects

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ABSTRACT

The purpose of this paper is to examine how any domain of Meinongian objects can be structured by a special kind of mereology. The basic definition of this mereology is the following: an object is part of another iff every characteristic property of the former is also a characteristic property of the latter. (The notions of *domain of Meinongian objects* and *characteristic property* will be carefully explained in the paper.) I will show that this kind of mereology ends up being very powerful for dealing with Meinongian objects. Mereological sums and products are not restricted in any way in a domain of Meinongian objects: there is a sum and a product for any pair of Meinongian objects. With the mereological operations of sum, product and complement, and two special Meinongian objects (a total object having every characteristic property and a null object having no characteristic property), we can define a full boolean algebra on Meinongian objects. Moreover, this kind of mereology is atomic and extensional: an atom is a Meinongian object having just one characteristic property and two objects are identical iff the same atoms are parts of both of them. A Meinongian object can finally be defined in mereological terms as the sum of the atoms of its characteristic properties.

Outline

(1) In the first section, a special notion of *part* is introduced. (2) In the second section, I will present a Meinongian axiomatic theory (a simplified version of Parsons' theory of nonexistent objects). (3) Using this theory as a framework, I will construct and study a mereological structure based on the special notion of

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part introduced in the first section. (4) Finally I will be presenting applications and extensions of this mereology in different fields (notably concerning the traditional round square). (5) As a conclusion I will outline a generalization by showing how this mereological structure can be constructed not only in the particular Meinongian theory I have been considering, but also in every domain of Meinongian objects.

1. Abstract Part

The mereology which I intend to present is based on a notion of *part* typically involved in sentences such as:

- (1) Rationality is a part of any human being.
- (2) Justice is a part of virtue.

What do we mean when we say that *x* is part of *y* in this special sense? Clearly it is different from what we mean when we say that fingers are parts of a hand, or the morning is a part of the day: in such cases (which are the most ordinary), *being a part of* means being *spatio-temporally included in*. Yet, justice is not spatio-temporally included in virtue, not even analogically.

It seems that what we mean by (1) and (2) has something to do with the instantiation of certain properties related to the notion of rationality, human beings, justice and virtue. Indeed, we could explain (1) by saying that everything that has the property of being human has also the property of being rational. And similarly, we could say that (2) expresses the fact that everything that has the property of being virtuous has also the property of being just.

This kind of use of part involving instantiation of properties (and how classes or bundles of properties are included in one another) has been studied by various authors.¹ The purpose of this paper is not to give an original account of this notion; it is rather to show its usefulness when applied to Meinongian objects.

If we assume that (1) and (2) express genuine mereological relations, we must clear up what kind of things are denoted by the terms *rationality*, *justice* and *virtue*. I will take them as designating a special sort of abstract objects: concepts of property.² The concept of virtue, for instance, or equivalently the

¹ See notably Goodman (1977) and Paul (2002). Tropes theorist also commonly use mereological machinery to describe how bundles of tropes are formed and are related to each others. Thus, studying this special sense of part is not a theoretical novelty.

² On the distinction between a property and its concept see Zalta (2000, p. 140 and ff.).

concept of *being virtuous*, contains every property implied by *being virtuous*: hence it contains the property of *being virtuous* itself, and others as (presumably) the properties of *being just*, *being courageous*, etc.

The mereological relations expressed in (1) and (2) could be understood like this: justice is a part of virtue because every property of justice (i.e., every property contained in the concept of *being just*) is also a property of virtue. Thus the following definition seems correct at first glance:

x is a part of y iff every property of x is also a property of y .

But rationality is an abstract object and thus we should assume that it has the property of *being abstract*, while no human being has the property of *being abstract* (since human beings are concrete objects). Hence, there is a property of rationality which is not a property of any human being, therefore rationality is not a part of any human being.

Of course, there seems to be something wrong in this argument: *being abstract* is a property of rationality, but not in the same way as *being rational* is a property of rationality; arguably, *being abstract* does not *belong* to the concept of rationality; but this concept (like every other concept) *is* abstract, hence rationality must exemplify somehow the property of *being abstract*.

This remark can be generalized to every abstract object. Let us say that every abstract object is *characterized* by some of its properties, and call them *characteristic properties*. The idea can be intuitively grasped by considering a few examples. The characteristic properties of a number like 2 are its mathematical properties (such as *being pair*, *being prime*, *being the successor of 1*, etc., but not properties like *being abstract*, *being colorless*, *being eternal*, etc.). More generally, the characteristic properties of a theoretical object are exactly the properties that the relevant theory attributes to this object. Similarly, the characteristic properties of a fictional object are exactly the properties that the relevant fiction attributes to this object (for example *being a detective* is a characteristic property of Sherlock Holmes, but not *being created by Conan Doyle*). For an intentional object, the characteristic properties are just those involved in the content of the representation; for example, if I am searching for a golden mountain, *being golden* and *being a mountain* are characteristic properties of the object I am searching for, while *not being existent* is a non-characteristic property.

It seems natural to assume that two abstract objects are identical iff they share exactly the same characteristic properties.

Should characteristic properties be identified with essential properties? No, if by essential properties we mean *necessary* properties (i.e., properties an object necessarily has): indeed, a number is necessarily an abstract object, hence *being abstract* is one of its essential properties, but it is not one of its characteristic properties (since *being abstract* is not a mathematical property).

We can now reformulate our definition of part:

x is a part of y iff every characteristic property of x is a characteristic property of y .

In terms of class of properties, this definition is equivalent to this: x is a part of y iff the class of the characteristic properties of x is included in the class of the characteristic properties of y .

According to this new definition the problem with (1) is resolved: although *being abstract* is a property of rationality, it is not one of its characteristic properties, so it is not relevant for parthood. Rationality is a part of human beings iff every *characteristic* property of rationality (i.e., every property implied by *being rational*) is a characteristic property of human beings.

Let us consider another example. What are the parts of the square in this special sense of part. By using the notion of spatio-temporal part, one may say that each side is a part of the square; but a side is not a part of the square according to our special definition, for a side has numerous characteristic properties the square does not have (for instance a side has the characteristic property of *being a segment* while the square has not this property). But the square has every characteristic property of a rhombus; therefore the rhombus is a part of the square. Moreover the rhombus has every characteristic property of the quadrilateral, thus the quadrilateral is a part of the rhombus. If one thinks that it is a very unusual way to talk about geometric figures, one may find this reformulation more appealing: the *concept* of rhombus is a part of the *concept* of square. One may also prefer to use *being included* or *being comprised*: the concept of rhombus is *included* or *comprised* in the concept of square. Whatever terms are finally used, the important aspect here is to have a clear view of the definition.

A last example: what are the parts of Sherlock Holmes? By using the ordinary notion of spatio-temporal part (in an analogical way, though), one may say that Sherlock Holmes' arm is a part of Sherlock Holmes. But it is clearly not a part in our special sense, since Sherlock Holmes' arm has characteristic properties Sherlock Holmes has not: for instance, Sherlock Holmes' arm may

have the characteristic property of *having a mass of less than 10kg*, while the whole Sherlock Holmes has not this property. However, concepts of properties such as rationality, intelligence, etc., are parts of Sherlock Holmes.

If we assume that Sherlock Holmes is the abstract object whose characteristic properties are exactly those ascribed to him in Conan Doyle's stories, and if we assume that Sherlock-Holmes-from-*The-Hound-of-the-Baskervilles* is the abstract object whose characteristic properties are exactly those ascribed to the protagonist of *The Hound of the Baskervilles*, then Sherlock-Holmes-from-*The-Hound-of-the-Baskervilles* is an abstract part of Sherlock Holmes. Indeed, every characteristic property of Sherlock-Holmes-from-*The-Hound-of-the-Baskervilles* will also be a characteristic property of the final Sherlock Holmes. We could say that Sherlock Holmes is the *mereological sum* of the Sherlock Holmes of each story.

In the following sections, I will show how this special kind of parthood can impose a structure on a domain of Meinongian objects.

2. Outline of a Meinongian Theory

2.1. Meinongian theories and characteristic properties

I will present a simplified version of Parsons' theory of nonexistent objects (see Parsons, 1980). Nevertheless, the mereology I will construct in this framework may be constructed similarly in any other Meinongian theory. For example, I constructed a first version of this mereology within Edward Zalta's theory of abstract objects. I am much indebted to Zalta (2000), an article in which a Leibnizian theory of concept is presented. An important part of the mereology presented in this paper can be seen as generalization and continuation of some ideas originally set forth in Zalta (2000).

We could say that Parsons' theory differ from Zalta's on the way they express the notion of *characteristic property*. According to Parsons, there is a distinction between two kinds of properties: some properties are nuclear, others are extranuclear. The former are the characteristic properties. According to Zalta, there is a distinction between two kinds of predication; exemplification and encoding. The properties an abstract object encodes are characteristic ones. The status of characteristic properties is thus very different according to Parsons' or Zalta's view. For Parsons, properties *themselves* are

characteristic or not. For Zalta, a property is not characteristic in itself; a property is characteristic because an object possesses it in a special way.

Anyway, both theories agree on the following principle (interpreted in two different ways whether the phrase *characteristic property* means *nuclear property* or *encoded property*):

For any condition φ on characteristic properties, there is an object which has exactly all the characteristic properties satisfying φ .

By a *domain of Meinongian objects* I mean a domain described by such a principle.³ In the last section of this paper, some general rules for endowing any domain of Meinongian objects (and in particular Zalta's domain of abstract object) with a mereological structure will be provided.

2.2. Theory M

I will now present a Meinongian theory M. The language of M is a second-order language with two distinguished kinds of predicates: nuclear predicates and extranuclear predicates. I will use '!' to distinguish extranuclear predicates.

The language of M consists of individual terms noted as usual a, b, c, \dots , (constants) and x, y, z, x_1, x_2, \dots (variables); n -adic nuclear predicates ($n \geq 1$) noted like standard predicate P^n, Q^n, R^n, \dots (constants) and F^n, G^n, H^n, \dots (variables); and n -adic extranuclear predicates ($n \geq 1$) noted similarly $P^n!, Q^n!, R^n!, \dots$ (constants) and $F^n!, G^n!, H^n!, \dots$ (variables). We skip the n 's for adicity whenever there is no ambiguity. I call monadic predicates *properties* and polyadic predicates *relations*. The other symbols of the language are: a distinguished extranuclear predicate $E!$, standards connectors $\neg, \&, \vee, \rightarrow, \equiv$, quantifiers \exists and \forall , and identity symbol: $=$.

The extranuclear property $E!$ plays an important role in the theory. According to Parsons, $E!$ is the property of *being existent*. For Zalta, whose theory refers to a similar property, it is the property of *being concrete*, or *being located in space and time*. I think it is somehow the same. Anyone can choose the metatheoretical term he prefers as long as he understand correctly the role that $E!$ will play in the theory. I will refer to it as the property of *being concrete*,

³ Concerning Graham Priest's (2005) theory of intentionality and Francesco Berto's (2011) modal Meinongianism, it is not clear if we have a domain of Meinongian objects: according to their view, for any condition there is an object satisfying precisely this condition *in a certain world*. If we identify the characteristic properties of this object with the properties this object owns *in this world*, then it seems that the domain of such objects can be deemed Meinongian in conformity to our definition.

and therefore objects lacking this property will be considered as *abstract* objects.

How one is supposed to know if a given predicate is a nuclear or an extranuclear one? We have no precise criterion. It is a weakness of Parsons' account as he has never cleared this point up, at least not in a satisfying way. This weakness is of no consequence to the mereology that I will construct afterwards. But in order to clarify as much as possible this distinction between nuclear and extranuclear predicates, let us give some indications. Ontological properties like *existing*, *being concrete*, logical properties like *being complete*, *being contradictory*, are expected to be extranuclear properties, as well as certain intentional properties and relations like *being thought by Meinong*. On the other hand, properties like *being red*, *being made of gold*, *being a mountain*, etc., are nuclear. (Nuclear property are likely to outnumber extranuclear properties by far.)

Formulas of the language are defined in the usual way. An atomic formula is a n -adic predicate $K(!)^n$ (nuclear or extranuclear) and n individual terms t_1, \dots, t_n :

$$K(!)^n t_1 \dots t_n$$

The other rules for identity, quantifiers and connectors are standard.

The theory M is an axiom system where standard first-order logic is complemented with elimination and introduction rules for quantifiers binding variable nuclear and extranuclear predicates (these rules are analogous to the rules for quantifiers binding individual variable), and two extra axioms.

The first axiom is:

$$(LL) \quad x = y \equiv \forall F(Fx \equiv Fy)$$

This axiom asserts that two objects are identical iff they share exactly the same *nuclear* properties. (Remember that F is a *nuclear* variable predicate.) Thus, this axiom can be understood as Leibniz Law restricted to nuclear predicates.

The last axioms are produced by the following axiom schema:

$$(OBJ) \quad \exists x \forall F(Fx \equiv \varphi) \quad \text{where } x \text{ is not free in } \varphi$$

This schema means that for any condition φ , there is an object having exactly every *nuclear* property satisfying φ . From this principle, we know a

priori that there is a domain of objects having any sorts of combination of nuclear properties, i.e., a Meinongian domain of objects.

Notably, for any finite class of nuclear properties P_1, \dots, P_n , there is an object having exactly those properties. Its existence is assured by this instance of (OBJ):

$$\exists x \forall F (Fx \equiv (F = P_1 \vee \dots \vee F = P_n))$$

The property of being concrete *E!* is extranuclear, so this axiom schema does not allow to prove that there is a concrete golden mountain, i.e., an object having the properties of *being a mountain*, *being golden* and *E!*. The two former properties are nuclear ones, but the latter is not. It is thus provable that there is a golden mountain, but not a *concrete* golden mountain.

2.3. Introducing definite descriptions

Let us use the notation $\exists! \alpha$ for *there is a unique α* :

$$(Df\exists!) \quad \exists! \alpha(\varphi) =_{df} \exists \alpha(\varphi \ \& \ \forall \beta(\varphi(\beta/\alpha) \rightarrow \beta = \alpha))$$

where $\varphi(\beta/\alpha)$ is the result of substituting every occurrence of α in φ by an occurrence of β .

With (LL) it is easy to show that an object whose existence is assured by an instance of (OBJ) is unique for any given condition φ . In other terms, what follows is a theorem schema:

$$(OBJ!) \quad \exists! x \forall F (Fx \equiv \varphi) \quad \text{where } x \text{ is not free in } \varphi$$

It will be useful to introduce a notation for this unique object determined by the fact that its nuclear properties are exactly those satisfying a condition φ . Resorting to Russell's iota notation, we get: $\iota x(\forall F(Fx \equiv \varphi))$. The theorem schema (OBJ!) assures that every description of this form is indeed satisfied by a unique object.

The unrestricted addition of definite descriptions brings with itself some important logical modification. For the sake of simplicity I will only allow the use of a definite description $\iota x(\varphi)$ under the condition that it has been already proved that this description φ is indeed satisfied by a unique object. Thus definite descriptions enter M thanks to the following rule:

If $\exists! x(\varphi)$ is a theorem, then we can use the complex individual term $\iota x(\varphi)$.

The schema (OBJ!) allows us to use terms of the form $\iota x(\forall F(Fx \equiv \varphi))$.

The logic for definite descriptions in theory M is very simple. The object $\iota x(\varphi)$ satisfies ψ iff there is an object satisfying both φ et ψ :

(DD) If ψ is a formula where the term $\iota x(\varphi)$ appears, the following is an axiom:

$$\psi \equiv \exists y(\varphi(y/x) \ \& \ \psi(y/\iota x(\varphi)))$$

3. Meinongian Mereology

3.1. Part and proper part

My aim is to construct a mereology based on this special definition of *part*:

x is a part of y iff every characteristic property of x is a characteristic property of y .

In the theory M, characteristic properties must be identified with the nuclear ones. The *part* relation, for which I will use the symbol \preceq , should thus be defined in the following way:

$$(Df\preceq) \quad x \preceq y =_{df} \forall F(Fx \rightarrow Fy)$$

It is easy to prove that this relation is reflexive, transitive and antisymmetric:⁴

$$(T1) \quad x \preceq x$$

$$(T2) \quad (x \preceq y \ \& \ y \preceq z) \rightarrow x \preceq z$$

$$(T3) \quad (x \preceq y \ \& \ y \preceq x) \rightarrow x = y$$

I define *proper part* in the usual way: x is a proper part of y iff x is a part of y distinct from y .

$$(Df<) \quad x < y =_{df} x \preceq y \ \& \ x \neq y$$

3.2. The null object and the total object

There is an object having no nuclear property. Let us call it the *null object*. I introduce the notation α_{\emptyset} defined by this definite description:⁵

⁴ The extent of this paper does not allow me to go into a thorough proof for the theorems, but I will give an outline for the most important and difficult ones.

$$(Df\emptyset) \quad \alpha_{\emptyset} =_{df} \lambda x (\forall F (Fx \equiv F \neq F))$$

There is an object having every nuclear property. Let us call it the *total object*. I introduce the notation α_{Ω} defined by this definite description:

$$(Df\Omega) \quad \alpha_{\Omega} =_{df} \lambda x (\forall F (Fx \equiv F = F))$$

Of course, any contradiction could be substituted to the condition $F \neq F$, and any tautology could be substituted to the condition $F = F$. No nuclear property F satisfies the condition $F \neq F$, hence α_{\emptyset} has no nuclear property. Every nuclear property F satisfies $F = F$, hence α_{Ω} has every nuclear property. Therefore, the following formulas are theorems:

$$(T4) \quad \neg F\alpha_{\emptyset}$$

$$(T5) \quad F\alpha_{\Omega}$$

The null object is not an object without any property at all. It is an object without *nuclear property*. One can consider this object as the concept of nothingness or nonbeing. The null object however has extranuclear properties; for instance it has the extranuclear property of *being abstract*.

Similarly, the total object has only every *nuclear* property but lacks many extranuclear properties; notably it lacks the property *E!*.

With (T4) and (T5) it is easy to prove that the null object is part of every object, and every object is a part of the total object.

$$(T6) \quad \alpha_{\emptyset} \leq x$$

$$(T7) \quad x \leq \alpha_{\Omega}$$

It is worth noting that the existential generalization of these theorems are the mereological principle known as *Bottom* and *Top*.

$$(Bottom) \quad \exists y \forall x y \leq x$$

$$(Top) \quad \exists y \forall x x \leq y$$

Two theorems follow respectively from (T6) and (T7): the null object is a proper part of every object except itself, and every object except the total object is a proper part of the total object.

$$(T8) \quad x \neq \alpha_{\emptyset} \rightarrow \alpha_{\emptyset} < x$$

⁵ Remind that the theorem schema (OBJ!) allows us to introduce this definite description as well as every others that will be of this form: $\lambda x (\forall F (Fx \equiv \varphi))$.

$$(T9) \quad x \neq o_{\Omega} \rightarrow x < o_{\Omega}$$

3.3. Atoms

An *atom* in standard mereology is defined as an object having no proper part. I have shown that the null object is a proper part of every object except itself (see (T8)); hence, according to this standard definition, the null object would be the single atom of our mereology.

But if I define an atom as an object having no proper part *except the null object*, the atoms would be objects having *only one nuclear property*. This notion of atom looks more promising from the start, and we will see beneath that these atoms are indeed the basic building blocks of every other object, except the null object. By contrast, the null object should not be considered as an atom since it does not play a genuine role in the constitution of other objects.

I will define atoms as follows:

$$(DfAtom!) \quad Atom!(x) =_{df} \exists ! F Fx$$

(About the use of ‘!’ in *Atom!*, see the note).⁶ An atom is thus an object having exactly one nuclear property. Equivalently, an atom is an object having a single proper part (the null object); an atom is an object whose parts are only itself and the null object. Those equivalences could be taken as definition as well as (Df*Atom!*).

$$(T10) \quad Atom!(x) \equiv \exists ! y y < x$$

$$(T11) \quad Atom!(x) \equiv \forall y (y \leq x \rightarrow (y = x \vee y = o_{\emptyset}))$$

For every nuclear property *F*, there is a unique atom having *F*:

$$(T12) \quad \exists ! x (Atom!(x) \ \& \ Fx)$$

This theorem allows us to introduce a definite description for the atom of the nuclear property *F*. I will note this term *a_F*:

$$(Dfa) \quad a_F =_{df} \iota x (Atom!(x) \ \& \ Fx)$$

⁶ The property *Atom!* would be a complex extranuclear property; but in order to keep the theory simple in this paper I have not introduced this kind of complex terms in the language. Hence, I only define the expression *Atom!(x)*. I note *Atom!(x)* and not simply *Atom(x)* because this notion should correspond to an extranuclear property; but ‘!’ has no real function here.

The nuclear property F and its atom a_F are related in interesting ways: two nuclear properties are identical iff their respective atoms are identical, and an object has a nuclear property iff its atoms is a part of the object:

$$(T13) \quad F = G \equiv a_F = a_G$$

$$(T14) \quad Fx \equiv a_F \leq x$$

This last theorem is very important: it shows that the predication of a nuclear property has a sort of *translation* in mereological terms.

3.4. Extensionality

Extensionality is an important aspect of parthood. The idea of extensionality can be informally expressed with the following principle: two composed objects (i.e., objects having proper parts) are identical iff they share exactly the same proper parts.

In standard mereology, extensionality is obtained by adding to the three basic principles (reflexivity, transitivity and antisymmetry of part relation) another principle, the principle of *strong supplementation*:⁷ if x is not a part of y then there is a part of x that does not overlap y .

The notion of *overlapping* is not yet defined, so the proof of this principle must wait. However I can already prove an atomistic version of this principle: if x is not a part of y then there is an atom z such that z is a part of x and is not a part of y .⁸

$$(T15) \quad \neg(x \leq y) \rightarrow \exists z(Atom!(z) \ \& \ z \leq x \ \& \ \neg(z \leq y))$$

Extensionality also can be proved in an atomistic version: two objects are identical iff exactly the same atoms are parts of both of them.⁹

$$(T16) \quad x = y \equiv \forall z(Atom!(z) \rightarrow (z \leq x \equiv z \leq y))$$

In other terms: atoms make the identity of object. (I will show later that every object is a sum of atoms, in a sense that will be defined.)

⁷ See Varzi (2012, 3.2).

⁸ Suppose the antecedent: $\neg(x \leq y)$. By (Df \leq) there is a nuclear property F such that $Fx \ \& \ \neg Fy$. By (T14), this formula is equivalent to $a_F \leq x \ \& \ \neg(a_F \leq y)$. Moreover it easy to prove that $Atom!(a_F)$. Hence, $\neg(x \leq y)$ implies $Atom!(a_F) \ \& \ a_F \leq x \ \& \ \neg(a_F \leq y)$, and by existential generalization we get to the consequent: $\exists z(Atom!(z) \ \& \ z \leq x \ \& \ \neg(z \leq y))$.

⁹ Here is a very brief and informal version of the proof. By (LL), two objects are identical iff they share exactly the same nuclear properties. By (T14), an object has a nuclear property iff the atom of this property is a part of this object. Therefore, x being identical to y is equivalent to x and y containing exactly the same atoms.

Extensionality can also be proved in an almost standard way: two *non-atomic* objects are identical iff they share exactly the same proper parts.¹⁰

$$(T17) \quad \neg Atom!(x) \rightarrow (x = y \equiv \forall z(z \leq x \equiv z \leq y))$$

In this mereology, *non-atomic object* is not equivalent to *object having proper parts* since every atom has the null object as proper part. That is why extensionality is not proved in the usual way but in a slightly modified version.

3.5. Overlapping

Overlapping is usually defined as *having a common part*. But since the null object is part of every object, this definition would have the consequence that any object overlaps any other. Overlapping would be a trivial relation, hence it must be defined in another way.

I will take this definition: two objects overlap each other iff they have a common nuclear property.

$$(DfO) \quad xOy =_{df} \exists F(Fx \& Fy)$$

Other equivalent definitions could be used: *x overlaps y* iff they have a common atom, or iff they have a common *non-null* part.

$$(T18) \quad xOy \equiv \exists z(Atom!(z) \& z \leq x \& z \leq y)$$

$$(T19) \quad xOy \equiv \exists z(z \neq o_\emptyset \& z \leq x \& z \leq y)$$

Strong supplementation can now be expressed and it is indeed a theorem: if *x* is not a part of *y* then there is a part of *x* that does not overlap *y*. (The proof is straightforward from (T15).)

$$(T20) \quad \neg(x \leq y) \rightarrow \exists z(z \leq x \& \neg zOy)$$

¹⁰ If *x* is not an atom, either *x* is the null object or *x* is an object containing at least two distinct atoms. If *x* is the null object, the formula is trivially true. If *x* is composed of at least two atoms, then suppose an object *y* identical to *x*. By (T16), *x* and *y* have exactly the same atoms as parts, and since *x* is composed of at least two atoms, those atoms are proper parts of *x* and *y*. From there, it is easy to show that *x* and *y* share exactly the same proper part. (You just have to suppose an arbitrary proper part *z* of *x* and show, using (Df<), (Df≤) and (T14), that it is also a proper part of *y*.)

3.6. Sum, product and complement. A full boolean algebra

3.6.1. Sum

I define the sum of x and y , noted $(x+y)$, as the unique object z having exactly all nuclear properties of x and y . (This definition is directly inspired by Zalta 2000.)

$$(Df+) \quad (x+y) =_{df} \iota z (\forall F (Fz \equiv Fx \vee Fy))$$

Sum is idempotent, commutative and associative:¹¹

$$(T21) \quad (x+x) = x$$

$$(T22) \quad (x+y) = (y+x)$$

$$(T23) \quad ((x+y)+z) = (x+(y+z))$$

In virtue of associativity, we may define sum of n terms:

$$(Df+n) \quad (x_1 + \dots + x_n) =_{df} (x_1 + (\dots + x_n))$$

$$(T24) \quad (x_1 + \dots + x_n) = \iota z (\forall F (Fz \equiv Fx_1 \vee \dots \vee Fx_n))$$

The null object is a neutral element and the total object is an absorbing element for this operation:

$$(T25) \quad (x+o_\emptyset) = x$$

$$(T26) \quad (x+o_\Omega) = o_\Omega$$

An object overlaps the sum of x and y iff it overlaps x or it overlaps y .

$$(T27) \quad zO(x+y) \equiv (zOx \vee zOy)$$

Sum in standard mereology (see Varzi 2012, 4.2) is defined as follows: the sum of x and y is the unique object z such that an object overlaps z iff this object

¹¹ Most of the proofs in this section use the following theorems schema: $(DD^*) \quad \iota x (\forall G (Gx \equiv \phi)) \equiv \phi(F/G)$ where $\phi(F/G)$ is the formula ϕ where F is substituted for G . Now, here is a proof of (T22). Assume that $(x+y)$ has an arbitrary property G . By (Df+). The formula $G(x+y)$ is equivalent to $G \iota z (\forall F (Fz \equiv Fx \vee Fy))$. By (DD^*) , it is equivalent to $Gx \vee Gy$. It is trivially equivalent to $Gy \vee Gx$. By (DD^*) again, it is equivalent to $G \iota z (\forall F (Fz \equiv Fy \vee Fx))$. And by (Df+) it is equivalent to $G(y+x)$. Thus, for an arbitrary G it is proved that $G(x+y) \equiv G(y+x)$. By (LL) it is now easy to prove $(x+y) = (y+x)$. The way we proceed in this proof clarify why sum has some of the logical properties of disjunction (*idem* for product and conjunction, and for complement and negation). It is worth noting also that most of the theorems of this sections correspond to tautologies of propositional logic. For instance (T25) corresponds to $(p \vee \perp) \equiv p$. (For more details about this kind of proof, see the last part of Zalta, 2000).

overlaps x or y . We can see that our notions of sum and overlapping are correctly defined since this standard definition is a theorem:

$$(T28) \quad \exists! z \mathcal{A} w(wOz \equiv (wOx \vee wOy))$$

$$(T29) \quad (x+y) = \iota z \mathcal{A} w(wOz \equiv (wOx \vee wOy))$$

The sum of any two objects exists (it is a consequence of (OBJ)): our mereology is committed to an unrestricted principle of composition.

3.6.2. Product

Product is similar to sum in many ways. The product of x and y , noted $(x \times y)$, is defined as the object z having exactly all the nuclear properties common to x and y .

$$(Df \times) \quad (x \times y) =_{df} \iota z (\forall F(Fz \equiv Fx \& Fy))$$

This operation is idempotent, commutative and associative:

$$(T30) \quad (x \times x) = x$$

$$(T31) \quad (x \times y) = (y \times x)$$

$$(T32) \quad ((x \times y) \times z) = (x \times (y \times z))$$

In virtue of associativity, we may define product of n terms:

$$(Df \times n) \quad (x_1 \times \dots \times x_n) =_{df} (x_1 \times (\dots \times x_n))$$

$$(T33) \quad (x_1 \times \dots \times x_n) = \iota z (\forall F(Fz \equiv Fx_1 \& \dots \& Fx_n))$$

The null object is an absorbing element and the total object is a neutral term for this operation:

$$(T34) \quad (x \times o_\emptyset) = o_\emptyset$$

$$(T35) \quad (x \times o_\Omega) = x$$

An object overlaps the product of x and y iff it overlaps both x and y .

$$(T36) \quad zO(x \times y) \equiv (zOx \& zOy)$$

Product in standard mereology is defined as follows: the product of x and y is the unique object z such an object overlaps z iff this object overlaps both x and y . As previously with the sum, we can see that our notions of product and overlapping are correctly defined since this standard definition is a theorem:

$$(T37) \quad \exists! z \mathcal{A} w(wOz \equiv (wOx \& wOy))$$

$$(T38) \quad (x+y) = \iota z \forall w (wOz \equiv (wOx \& wOy))$$

The product of any two objects exists (as a consequence of (OBJ!)). Even if two objects do not overlap each other, there is a product of them: the null object. Thus, we have the following theorem: two objects overlap each other iff their product is not the null object.

$$(T39) \quad xOy \equiv (x \times y) \neq o_\emptyset$$

It is worth noting that overlapping can be defined in at least four different ways. Indeed, x overlaps y iff:

- i) x and y have a common nuclear property
- ii) x and y contain a common atom
- iii) x and y have a common non-null part
- iv) the product of x and y is not null

3.6.3. Complement

The complement of x , noted $(-x)$, is the object having exactly all the nuclear properties x does not have.

$$(Df-) \quad (-x) =_{df} \iota y (\forall F (Fy \equiv \neg Fx))$$

The complement of the complement of x is x .

$$(T40) \quad (-(-x)) = x$$

An object overlaps the complement of x iff it is a non-null object that does not overlap x .

$$(T41) \quad yO(-x) \equiv (y \neq o_\emptyset \& \neg yOx)$$

There is a unique object y such that any object overlapping y is a non-null object that does not overlap x , and this unique object y is the complement of x .

$$(T42) \quad \exists! y \forall z (zOy \equiv (z \neq o_\emptyset \& \neg zOx))$$

$$(T43) \quad (-x) = \iota y (\forall z (zOy \equiv (z \neq o_\emptyset \& \neg zOx)))$$

Note the difference between the two following formulas: $\neg Fx$ and $F(-x)$. The former means that x is not an F , the latter means that the complement of x is an F . However those formulas are equivalent:

$$(T44) \quad \neg Fx \equiv F(-x)$$

3.6.4. A full boolean algebra

The operations of sum, product and complement, along with the null object α_0 and the total object α_Ω , produce a full boolean algebra. This result is not surprising since it appears very clearly that those three operations, sum, product and complement, have respectively the logical properties of disjunction, conjunction and negation, and the null object and the total object play respectively the role of contradiction and tautology.

I have already shown that the operations of sum and product are commutative and associative. They are also distributive for each other in the following way:

$$(T45) \quad (x+(y \times z)) = ((x+y) \times (x+z))$$

$$(T46) \quad (x \times (y+z)) = ((x \times y) + (x \times z))$$

There are also the following identities characterizing a boolean algebra:

$$(T47) \quad (x+(x \times y)) = x$$

$$(T48) \quad (x \times (x+y)) = x$$

$$(T49) \quad (x+(-x)) = \alpha_\Omega$$

$$(T50) \quad (x \times (-x)) = \alpha_0$$

Those identities are easily proved (with the method explained in note 11). About mereology and boolean algebra, see Pontow & Schubert (2006). Without the null object, we would obtain an incomplete boolean algebra. Mereology is generally suspicious about the existence of an object that is part of every object. It is an interesting feature of this mereology to assure the existence of this object and to make clear what it is: it is simply an object whose description in terms of nuclear property is to have no nuclear property at all. (Remind that it is different from not having property at all; the null object has extranuclear properties like any other object.)

3.7. General sum

I have shown that two objects are identical iff they have exactly the same atoms as parts (see (T16)). I want now to express the fact that all objects are made of atoms. In more rigorous terms, I will prove that every object is the sum of the atoms of its nuclear properties.

The problem is that the notion of sum that I have defined for the moment does not allow us to express this idea. I must introduce the notion of *general sum*.

The following theorems schema is an instance of (OBJ!):

$$(T51) \quad \exists !x \forall F (Fx \equiv \exists y (\varphi \ \& \ Fy))$$

In less formal terms, for any condition φ , there is a unique object having exactly every nuclear property of every object satisfying φ . This unique object is *the sum of x 's such that φ* , and I introduce the following notation for it:

$$(Df\sigma) \quad \sigma x(\varphi) =_{df} \iota y (\forall F (Fy \equiv \exists x (\varphi \ \& \ Fx)))$$

The general sum $\sigma x(\varphi)$ and the sum $(x_1 + \dots + x_n)$ are identical iff the objects x_1, \dots, x_n are exactly all objects satisfying φ .¹²

$$(T52) \quad \sigma x(\varphi) = (x_1 + \dots + x_n) \equiv \\ (\varphi(x_1/x) \ \& \ \dots \ \& \ \varphi(x_n/x) \ \& \ \forall y (\varphi(y/x) \rightarrow (y = x_1 \vee \dots \vee y = x_n)))$$

This theorem confirms that the notion of general sum is correctly defined. The general sum of objects such that φ is indeed the sum of every object satisfying φ .

The standard way to define general sum in mereology is the following:¹³ the sum of x 's such that φ is the unique object y such that every object overlapping y overlaps at least one object satisfying φ . This definition is provably equivalent:

$$(T53) \quad \exists !y \forall z (yOz \equiv \exists x (\varphi \ \& \ xOz))$$

$$(T54) \quad \sigma x(\varphi) = \iota y (\forall z (yOz \equiv \exists x (\varphi \ \& \ xOz)))$$

Our resources allow us to express as a theorem the claim that every object is the sum of the atoms of its nuclear properties:¹⁴

¹² From right to left. By (LL), the formula $\sigma x(\varphi) = (x_1 + \dots + x_n)$ is equivalent to $F(\sigma x(\varphi)) \equiv F(x_1 + \dots + x_n)$. By (DD*) , (Df σ) and (Df+ n), we prove that this formula is equivalent to $\exists x (\varphi \ \& \ Fx) \equiv (Fx_1 \vee \dots \vee Fx_n)$. In other terms, there is an x satisfying φ and having an arbitrary property F iff one of the x_1, \dots, x_n has this arbitrary property F . It is clear that this formula implies that only the x_1, \dots, x_n satisfy φ . – From left to right. Assume that the objects x_1, \dots, x_n are all objects satisfying φ . Then, for an arbitrary property F , there is an object satisfying φ and having F iff one of the x_1, \dots, x_n has F . That is: $\exists x (\varphi \ \& \ Fx) \equiv (Fx_1 \vee \dots \vee Fx_n)$. And now by (DD*), (Df σ), (Df+ n) and (LL) we show that this formula is equivalent to $\sigma x(\varphi) = (x_1 + \dots + x_n)$.

¹³ See Varzi (2012, 4.4). In Hovda (2009), the definition I mention corresponds to fusion of type 1.
¹⁴ As previously (see note 12), we prove by (LL) that $x = \sigma y(\varphi)$ is equivalent to $Gx \equiv G(\sigma y(\varphi))$. By (DD*) and (Df σ), we prove that this formula is equivalent to $Gx \equiv \exists y \exists F (Fx \ \& \ y = a_F \ \& \ Cy)$. All we have to do now is to prove this formula. – From left to right. The formula Gx trivially implies the formula $Gx \ \& \ a_C = a_C \ \& \ G a_C$ (since $a_C = a_C$ and $G a_C$ are obvious theorems). Then, using existential

$$(T55) \quad x = \sigma y (\exists F (F x \& y = a_F))$$

This result shows that atoms play a role of composition *vis-à-vis* all other objects. An object is nothing but a sum of atoms: the sum of the atoms of its nuclear properties.

4. Applied Mereology

I have so far dealt with the most general features of the Meinongian mereology. In this last part, I will consider two applications on more specific fields. I will show how the theory deals with:

1. The notion of concept of property (and other cognates notions)
2. The notions of contradictory object and incomplete object

My purpose in this section is to illustrate how the mereology I have constructed increases the expressive power of the Meinongian theory.

4.1. Concepts of property

Consider the two examples I started from:

- (1) Rationality is a part of all human beings.
- (2) Justice is a part of virtue.

How can we express those sentences? I assume that justice, virtue and rationality are concepts of property. The concept of a nuclear property F can be defined in theory M as the object whose nuclear properties are those properties *implied by* F . This notion of implication for a property could be defined (with modality for example)¹⁵, but here for the benefit of simplicity I will take it as a primitive one.

generalization twice, we get to the formula $\exists y \exists F (F x \& y = a_F \& C y)$. – From right to left. Let b be an arbitrary instance of y and P an arbitrary instance of F . We get to the formula $P x \& b = a_P \& C b$. The last two conjuncts entail $C a_P$, and since an atom has only one nuclear property, it entails $C = P$. Therefore the first conjunct $P x$ entails $C x$.

¹⁵ For example: F implies G iff necessarily every concrete object having F also has G . More formally, it would be expressed in that way: $F \Rightarrow G =_{\text{df}} \Box \forall x (E! x \rightarrow (F x \rightarrow G x))$. But I have not introduced modal operators in the theory M , therefore I cannot use this definition.

4.1.1. Definition of concept of property

I will use the notation $F \Rightarrow G$ for *the nuclear property F implies the nuclear property G*. To simplify things, let us assume that this notion of implication is introduced only for nuclear properties.

I will admit only few principles about this relation of implication. First: it is reflexive and transitive. Second: if a concrete object has a nuclear property F , then it also has every nuclear properties implied by F .

$$\begin{aligned} (A \Rightarrow 1) \quad & F \Rightarrow F \\ & (F \Rightarrow G \ \& \ G \Rightarrow H) \rightarrow F \Rightarrow H \\ (A \Rightarrow 2) \quad & E!x \rightarrow (Fx \rightarrow \forall G((F \Rightarrow G) \rightarrow Gx)) \end{aligned}$$

(The restriction on concrete objects in (A⇒2) is a consequence of (OBJ): if we had assumed that an object having P also has every nuclear property P implies, there would presumably be instances of (OBJ) in contradiction with it, since (OBJ) assures that there is an object having P and no other nuclear property. As I will explain later, (A⇒2) only asserts that concrete objects are expected to be coherent. That seems acceptable.)

For every nuclear property F there is a concept of F , i.e., a unique x having exactly all the nuclear properties implied by F . I will note this object: c_F .

$$\begin{aligned} (T56) \quad & \exists !x \forall G(Gx \equiv F \Rightarrow G) \\ (Dfc) \quad & c_F = \iota x(\forall G(Gx \equiv F \Rightarrow G)) \end{aligned}$$

4.1.2. Formalizing (1) and (2)

Let *Just* and *Virtuous* be respectively the nuclear properties of *being just* and *being virtuous*, the sentence (2) can be represented in the following way:

$$(2') \quad c_{Just} \preceq c_{Virtuous}$$

This sentence is true iff being virtuous implies being just:

$$c_{Just} \preceq c_{Virtuous} \equiv Virtuous \Rightarrow Just$$

This formula is a theorem if we substitute the variables F and G for *Just* and *Virtuous*: the concept of F is part of the concept of G iff G implies F .

$$(T57) \quad c_F \preceq c_G \equiv F \Rightarrow G$$

Let us take the nuclear properties *Rational* and *Human*. The sentence (1) could be similarly expressed like this:

$$(1?) \quad c_{Rational} \preceq c_{Human}$$

But this seems to mean that rationality is a part of humanity (i.e., the concept of *being human*), not that it is part of *all human beings*. Hence (1) should rather be expressed like this:

$$(1') \quad Human(x) \rightarrow c_{Rational} \preceq x$$

This formula is true iff *being human* implies *being rational*:

$$(Human(x) \rightarrow c_{Rational} \preceq x) \equiv Human \Rightarrow Rational$$

More generally, an object having the nuclear property *F* has the concept of the nuclear property *G* as part iff *F* implies *G*.

$$(T58) \quad (Fx \rightarrow c_G \preceq x) \equiv F \Rightarrow G$$

4.1.3. Other theorems about concepts of property

The concept of *F* is the sum of the atoms of the properties implied by *F*.

$$(T59) \quad c_F = \sigma x (\exists G (F \Rightarrow G) \ \& \ x = a_G)$$

If the concept of *F* has a nuclear property *G* then the concept of *G* is a part of the concept of *F*. (For example the concept of square has the nuclear property of *being a rhombus*, therefore the concept of rhombus is a part of the concept of square.)

$$(T60) \quad Gc_F \rightarrow c_G \preceq c_F$$

If the concept of *F* is a part of an object *x*, then the concept of every nuclear property implied by *F* is also a part of *x*. (For example, the concept of rhombus is a part of the concept of square, and since *being a rhombus* implies *being a quadrilateral*, the concept of quadrilateral is also a part of the concept of square.)

$$(T61) \quad c_F \preceq x \rightarrow \forall G (F \Rightarrow G) \rightarrow c_G \preceq x$$

The null object is not the concept of any nuclear property (since any nuclear property at least implies itself, by $(A \Rightarrow 1)$):

$$(T62) \quad \neg \exists F c_F = o_\emptyset$$

There may be a nuclear property whose concept is the total object: it would be a nuclear property implying every nuclear property.

$$(T63) \quad c_F = o_\Omega \equiv \forall G (F \Rightarrow G)$$

Two concepts of nuclear properties overlap iff their product contains at least one concept of nuclear property;¹⁶

$$(T64) \quad c_F O c_G \equiv \exists H (c_H \leq (c_F \times c_G))$$

4.1.4. Coherent objects

I will define an object as *coherent* iff for every nuclear property F if this object has F then this object also has every property implied by F . Equivalently, that means that an object is coherent iff for every nuclear property F such that its atom is part of this object, then the concept of F is also a part of this object.

$$(Df\textit{Coherent}!) \quad \textit{Coherent}!(x) =_{df} \forall F (F x \rightarrow \forall G ((F \Rightarrow G) \rightarrow G x))$$

$$(T65) \quad \textit{Coherent}!(x) \equiv \forall F (a_F \leq x \rightarrow c_F \leq x)$$

(*Coherent!* is a defined expression like *Atom!*. The ‘!’ has no real function here. See note 6.)

The following objects are coherent: the null object, the total object, concrete objects and every concept of property.

$$(T66) \quad \textit{Coherent}!(o_\emptyset)$$

$$(T67) \quad \textit{Coherent}!(o_\Omega)$$

$$(T68) \quad E!x \rightarrow \textit{Coherent}!(x)$$

$$(T69) \quad \textit{Coherent}!(c_F)$$

If a coherent object has the nuclear property F then the concept of F is a part of this object.

¹⁶ This theorem is less obvious than the others. Here is a sketch of a proof. – From left to right. The concept of H is distinct from the null object (by (T62)). Therefore a non-null object is part of $(c_F \times c_G)$, which means that c_F overlaps c_G (by (T39)). – From right to left. Suppose that $c_F O c_G$. It means (by (DfO)) that there is a nuclear property P such that $P c_F$ and $P c_G$. By (Dfc), we know that $F \Rightarrow P$ and $G \Rightarrow P$. Suppose now that Q is an arbitrary nuclear property such that $Q c_H$. By (Dfc) we have $P \Rightarrow Q$. By transitivity (as I assumed in axioms (A \Rightarrow 1)) we also have $F \Rightarrow Q$ and $G \Rightarrow Q$. By (Dfc) we can infer $Q c_F$ and $Q c_G$. Thus, an arbitrary nuclear property of c_F is also a property of c_F and c_G . By universal generalization and by (Df \leq), we get to $c_F \leq c_F$ and $c_F \leq c_G$, from which it is easy to prove the formula $c_F \leq (c_F \times c_G)$, and by existential generalization we get finally to $\exists H (c_H \leq (c_F \times c_G))$.

$$(T70) \text{ Coherent!}(x) \rightarrow (F_x \rightarrow c_F \preceq x)$$

For coherent objects thus, having a nuclear property can be “translated” in mereological terms as the fact that the concept of this property is part of the object. It is worth noting the similitude between this theorem and (T14) according to which the predication of nuclear property can be “translated” as the fact that the atom of this property is a part of the object.

A coherent object thus can be understood as the sum of the concepts of its nuclear properties.

$$(T71) \text{ Coherent!}(x) \rightarrow x = \sigma y (\exists F (F_x \& y = c_F))$$

Note again the similitude between this theorem and (T55) according to which every object is the sum of the atoms of its nuclear properties.

And we can prove a “conceptual” version of extensionality for coherent objects: if x is a coherent object and is not a concept of property, then x is identical to an object y iff both x and y have exactly the same concepts as proper parts.

$$(T72) (\text{Coherent!}(x) \& \neg \exists F x = c_F) \rightarrow (x = y \equiv \forall z \exists F (z = c_F \rightarrow (z < x \equiv z < y)))$$

Two other interesting theorems: if an object is a sum of coherent objects then this object also is coherent, and *idem* for the product of coherent objects.¹⁷

$$(T73) (x = (x_1 + \dots + x_n) \& \text{Coherent!}(x_1) \& \dots \& \text{Coherent!}(x_n)) \rightarrow \text{Coherent}(x)$$

$$(T74) (x = (x_1 \times \dots \times x_n) \& \text{Coherent!}(x_1) \& \dots \& \text{Coherent!}(x_n)) \rightarrow \text{Coherent}(x)$$

4.1.5. Formalizing the round square

What about the well-known round square? How can we deal with it in our theory? Let us take the following nuclear properties: *Round*, *Square*, *Curved*,

¹⁷ It is obvious for the sum, but maybe it is less intuitive for the product. Here is a sketch of a proof. Suppose that two object x and y are coherent. If their product is null, then their product is coherent since the null object is coherent. If their product is not null, then they share nuclear properties. Suppose that F is one of those nuclear properties. It is easy to prove by (T70) that the concept of F is part of x and y (since x and y are coherent). Therefore the concept of F is a part of the product $(x \times y)$. Therefore the product $(x \times y)$ is a sum of concept, and that is a coherent object.

Four-sided, *Axially-symmetric*, and *Red*. I assume that both *Round* and *Square* imply *Axially-symmetric*; *Round* implies *Curved* and does not imply *Four-sided*; *Square* implies *Four-sided* and does not imply *Curved*; and neither *Round* nor *Square* implies *Red*.

The round square could be represented as the sum of the atom of *Round* and the atom of *Square*.

$$o_1 = (a_{Round} + a_{Square}) \\ Round(o_1) \ \& \ Square(o_1) \ \& \ \neg Curved(o_1) \ \& \ \neg Four-sided(o_1) \ \& \ \neg Axially-symmetric(o_1) \ \& \ \neg Red(o_1)$$

This round square is very minimal: it is round and square and nothing else.

A more interesting round square is the sum of the *concept* of *Round* and the *concept* of *Square*.

$$o_2 = (c_{Round} + c_{Square}) \\ Round(o_2) \ \& \ Square(o_2) \ \& \ Curved(o_2) \ \& \ Four-sided(o_2) \ \& \ Axially-symmetric(o_2) \ \& \ \neg Red(o_2)$$

The round square o_2 is not only round and square, but it has also every property implied by *Round* or by *Square* (or both). Note that it does not have any property whatsoever: in particular, it is not red since *Red* is not implied by *Round* neither by *Square*. It is worth noting that as a sum of concepts, o_2 is a coherent object. This round square is coherently round and square (though it is a contradictory object, as we will see later; on the contrary o_1 is not a contradictory object but it is not coherent).

There is also an interesting intermediate solution:

$$o_3 = (a_{Round} + a_{Square} + (c_{Round} \times c_{Square})) \\ Round(o_3) \ \& \ Square(o_3) \ \& \ Axially-symmetric(o_3) \ \& \ \neg Curved(o_3) \ \& \ \neg Four-sided(o_3) \ \& \ \neg Red(o_3)$$

This round square o_3 is round and square and has all the properties common to the concepts of *Round* and *Square*, therefore it is axially symmetric (since both *Round* and *Square* imply *Axially-symmetric*), but it is not four-sided nor curved.

We could also consider that the round square is a square having *in addition* the property of *being round*, and that is different from the square round which is a round having *in addition* the property of *being square*. Though this intuition seems obscure at first glance, we can give a clear account of it. The

first is the sum of the atom of *Round* and the concept of *Square*, and the second is the sum of the atom of *Square* and the concept of *Round*:

$$o_4 = (a_{Round} + c_{Square}) \\ Round(o_4) \ \& \ Square(o_4) \ \& \ Axially-symmetric(o_4) \ \& \ Four-sided(o_4) \ \& \\ \neg Curved(o_4) \ \& \ \neg Red(o_4)$$

$$o_5 = (a_{Square} + c_{Round}) \\ Round(o_5) \ \& \ Square(o_5) \ \& \ Axially-symmetric(o_5) \ \& \ Curved(o_5) \ \& \\ \neg Four-sided(o_5) \ \& \ \neg Red(o_5)$$

Those two objects are indeed distinct. The first is coherently a square (so it is four-sided and axially symmetric), and it has just in addition the property of *being round* (but only this one, therefore it is not curved). The second is coherently a round (so it is curved and axially symmetric), and it has just in addition the property of *being square* (but only this one, therefore it is not four-sided).

Maybe other sorts of round square can be defined but o_1 , o_2 , o_3 and o_4 are presumably the most interesting ones (I do not include o_5 since it would be a *square round* rather than a round square).

Note that they can be ordered by the proper part relation in the following way:

$$o_1 < o_3 < o_4 < o_2$$

I will say more about those objects in the next subsection when the notions of contradictory objects and incomplete objects will be defined.

4.2. Contradictory objects and incomplete objects

Contradictory and incomplete objects are among the most interesting fields of application (and the best-known for sure) for Meinongian theories. However in our Meinongian theory M we cannot give a satisfying definition of the notions of contradictory object and incomplete object (though we can talk about presumably contradictory objects like the round square). I must extend the theory in such a way that those notions will become definable. Then I will study them from a mereological perspective.

4.2.1. Negative properties

Complex properties could be introduced through operators of abstraction, but, since here I am going to deal only with negative nuclear properties, a less complex method is at disposal.

If F is a nuclear property then $non-F$ is a nuclear property. I will call this property a *negative property*. It is the *negation of F* .

How negative properties are expected to work? A rather natural idea at first glance is that having the property $non-F$ is equivalent to not having the property F .

Now, suppose that we take this as an axiom:

$$(Neg?) \quad non-Fx \equiv \neg Fx$$

This axiom raises a serious problem. For any property F , there is an instance of (OBJ!) according to which there is a unique object having exactly the two nuclear properties F and $non-F$. In mereological terms, it is the sum of the atoms of F and $non-F$. It is a theorem that this object has both F and $non-F$.

$$(T75) \quad F(a_{F+} a_{non-F}) \& non-F(a_{F+} a_{non-F})$$

It is important to realize that this formula *is not a contradiction*: it is not a formula of the form ‘ $\varphi \& \neg\varphi$ ’. But by (Neg?), this formula entails indeed the plain contradiction:

$$F(a_{F+} a_{non-F}) \& \neg F(a_{F+} a_{non-F})$$

If (Neg?) is an axiom of the theory, the whole axiomatic collapses into contradiction. Therefore it must be rejected.

However, the equivalence between $non-F$ and $\neg F$ seems relevant until a certain point. We should not entirely reject it but only restrict it. The most natural restriction is to consider that the equivalence stands only for concrete objects.

Thus I will take the following as an axiom: a concrete object has the nuclear property $non-F$ iff this object has not the nuclear property F .

$$(Neg) \quad E!x \rightarrow (non-Fx \equiv \neg Fx)$$

By (T75) and (Neg), I can no longer infer a contradiction but only the following theorem: the sum of the atoms of F and $non-F$ is not concrete.

$$(T76) \quad \neg E!(a_{F+} a_{non-F})$$

It seems very acceptable. More generally, (Neg) implies that if an object has both the nuclear properties F and $non-F$ or if it lacks both of them, it is not a concrete object.

$$(T77) (Fx \& non-Fx) \rightarrow \neg E!x$$

$$(T78) (\neg Fx \& \neg non-Fx) \rightarrow \neg E!x$$

Perhaps I should also take the identity of $non-non-F$ and F as an axiom. This principle seems to preserve theory from an undesirable multiplication of nuclear properties of the form: $non-non-non-...non-F$. But maybe one could argue against this principle, and anyway it does not play any role in what follows, so I will remain neutral.

4.2.2. A brief excursus on the equivalence between non-F and $\neg F$

It is worth considering the axiom (Neg) from a more general perspective. This axiom asserts that the equivalence between *having the characteristic property of not being F* and *not having the characteristic property of being F* does not stand for abstract object. I will take a few intuitive examples to illustrate this idea.

The number two has not the characteristic property of *being red*. But it seems unacceptable to attribute to this number the property of *not being red*. The characteristic properties of a number are expected to be mathematical properties.

Similarly, if we assume that Conan Doyle had never mentioned anything about a mole on Sherlock Holmes' left shoulder, it is true that Sherlock Holmes lacks the characteristic property of *having a mole on the left shoulder*. But Sherlock Holmes surely does not have the characteristic property of *not having a mole on the left shoulder*: in contradiction with my assumption, this would mean that Conan Doyle has indeed mentioned the absence of a mole on Sherlock Holmes' left shoulder (since characteristic property of a fictional objects are precisely those properties ascribed to the object by the relevant fiction).

This equivalence is indeed unacceptable for abstract object. As a consequence, (Neg) should not be extended to abstract objects.

4.2.3. Contradictory objects and complete objects

With negative nuclear properties, it is now easy to properly express the notion of contradictory object and incomplete object.

An object is *contradictory* iff there is at least a nuclear property F such that this object has both F and $non-F$.

$$(Df\textit{Contradictory!}) \quad \textit{Contradictory!}(x) =_{df} \exists F(Fx \& non-Fx)$$

An object is *complete* iff for all nuclear property F this object has F or $non-F$ (or both).

$$(Df\textit{Complete!}) \quad \textit{Complete!}(x) =_{df} \forall F(Fx \vee non-Fx)$$

As a consequence of these definitions and (T77) and (T78), contradictory objects and incomplete objects are not concrete objects:

$$(T79) \quad \textit{Contradictory!}(x) \rightarrow \neg E!x$$

$$(T80) \quad \neg \textit{Complete!}(x) \rightarrow \neg E!x$$

Thus a concrete object must be non-contradictory and complete.

$$(T81) \quad E!x \rightarrow (\neg \textit{Contradictory!}(x) \& \textit{Complete!}(x))$$

Although the null object seems somehow related to contradiction, it is not a contradictory object: it is a non-contradictory and incomplete object. On the other extremity, the total object is both contradictory and complete.

$$(T82) \quad \neg \textit{Contradictory!}(o_\emptyset) \& \neg \textit{Complete!}(o_\emptyset)$$

$$(T83) \quad \textit{Contradictory!}(o_\Omega) \& \textit{Complete!}(o_\Omega)$$

An object is contradictory iff its complement is incomplete, and an object is incomplete iff its complement is contradictory.

$$(T84) \quad \textit{Contradictory!}(x) \equiv \neg \textit{Complete!}(\neg x)$$

$$(T85) \quad \neg \textit{Complete!}(x) \equiv \textit{Contradictory!}(\neg x)$$

A sum of objects is contradictory if it contains a contradictory term; a product of object is non-contradictory if it contains a non-contradictory terms.

Similarly a sum of objects is complete if it contains a complete term; a product of object is incomplete if it contains an incomplete terms.

$$(T86) \quad \textit{Contradictory!}(x) \rightarrow \textit{Contradictory!}(x+y)$$

$$(T87) \quad \neg \textit{Contradictory!}(x) \rightarrow \neg \textit{Contradictory!}(x \times y)$$

$$(T88) \text{ Complete!}(x) \rightarrow \text{Complete!}(x+y)$$

$$(T89) \neg \text{Complete!}(x) \rightarrow \neg \text{Complete!}(x \times y)$$

4.2.4. Perfect objects

I define a *perfect object* as an object such that for every nuclear property F this object has either F or $\text{non-}F$ (but not both).

$$(Df \text{ Perfect!}) \text{ Perfect!}(x) =_{df} \forall F (\neg Fx \equiv \text{non-}Fx)$$

Perfect objects are in fact both complete and non-contradictory, like concrete objects.

$$(T90) \text{ Perfect!}(x) \equiv (\text{Complete!}(x) \ \& \ \neg \text{Contradictory!}(x))$$

$$(T91) E!x \rightarrow \text{Perfect!}(x)$$

Perfect objects have incomplete non-contradictory objects as proper parts and they are proper parts of contradictory complete objects:

$$(T92) (E!x \ \& \ y < x) \rightarrow (\neg \text{Contradictory!}(y) \ \& \ \neg \text{Complete!}(y))$$

$$(T93) (E!x \ \& \ x < y) \rightarrow (\text{Contradictory!}(y) \ \& \ \text{Complete!}(y))$$

In other terms we could say that perfect objects are *minimally complete* and *maximally non-contradictory*: a perfect object with one nuclear property less gives an incomplete object, and a perfect object with one nuclear property more gives a contradictory object.

There are other interesting mereological theorems about perfect objects, for instance: the complement of a perfect object is a perfect object, and two distinct perfect objects are such that their sum is a contradictory object and their product is an incomplete object.

$$(T94) \text{ Perfect!}(x) \rightarrow \text{Perfect!}(\neg x)$$

$$(T95) (\text{Perfect!}(x) \ \& \ \text{Perfect!}(y) \ \& \ x \neq y) \rightarrow \\ (\text{Contradictory!}(x+y) \ \& \ \neg \text{Complete!}(x \times y))$$

Note that the perfection of an object does not imply its coherence (following the definition that I gave for those notions): there may be incoherent perfect objects. Let us say a little more about coherence now.

4.2.5. Perfection and coherence. A definition of possible objects

I have not mentioned any axioms governing negative properties in relation to our primitive notion of implication for nuclear properties. For example, should I take as an axiom that no nuclear property implies both F and $non-F$? Consequently, every concept would be non-contradictory. – It is not a very clear matter and I prefer to remain neutral about this. Hence I will not assume any additional axioms in what follows. However, even without any assumption of this kind, there are some interesting theorems about coherence in relation to complete objects, contradictory objects and perfect objects.

If an object is coherent, then it is complete iff for every nuclear property F , the concept of F or the concept of $non-F$ (or both) is part of this object.

$$(T96) \text{ Coherent!}(x) \rightarrow (\text{Complete!}(x) \equiv \forall F(c_F \leq x \vee c_{non-F} \leq x))$$

If an object is coherent, then it is contradictory iff there is a nuclear property F such that the concept of F and the concept of $non-F$ are both parts of this object.

$$(T97) \text{ Coherent!}(x) \rightarrow (\text{Contradictory!}(x) \equiv \exists F(c_F \leq x \ \& \ c_{non-F} \leq x))$$

The total object for instance is a coherent complete contradictory object, and the null object is a coherent incomplete non-contradictory object.

If an object is coherent, then it is perfect iff for every property F either the concept of F or the concept of $non-F$ (but not both) is part of this object.

$$(T98) \text{ Coherent!}(x) \rightarrow (\text{Perfect!}(x) \equiv \forall F(\neg(c_F \leq x) \equiv c_{non-F} \leq x))$$

Concrete objects are coherent perfect objects. It can also be assumed that merely possible objects, like the golden mountain, are coherent perfect object.

This could be taken as a minimal definition of possible object.

$$(Df\text{Possible!}) \text{ Possible!}(x) =_{df} \text{Perfect!}(x) \ \& \ \text{Coherent}(x)$$

$$(T99) \text{ E!}x \rightarrow \text{Possible!}(x)$$

4.2.6. What about the round square?

Let us take the same nuclear properties we took in 4.5.1.: *Round*, *Square*, *Curved*, *Four-sided*, *Axially-symmetric*, and *Red*. Note that we now have in addition the negations of these nuclear properties. Let us assume that: *Round*

implies *Curved*, *Axially-symmetric*, *non-Square* and *non-Four-sided* and does not imply any other properties; *Square* implies *Four-sided*, *Axially-symmetric*, *non-Round* and *non-Curved* and does not imply any other property.

Recall that I distinguished four candidates for the round square:

$$\begin{aligned} o_1 &= (a_{Round}^+ a_{Square}) \\ o_3 &= (a_{Round}^+ a_{Square}^+ (C_{Round} \times C_{Square})) \\ o_4 &= (a_{Round}^+ C_{Square}) \\ o_2 &= (C_{Round}^+ C_{Square}) \end{aligned}$$

All four objects are *incomplete* relatively to the nuclear property *Red*: they are not *Red* nor *non-Red*.

The minimal round square o_1 is not a contradictory object: it is round and square and nothing else, therefore it is neither *non-Round* nor *non-Square*. It is an *incoherent non-contradictory object*.

The round square o_3 is also an incoherent non-contradictory object. But it is richer than o_1 since it is not only *Round* and *Square* but also *Axially-symmetric*. It would have more generally every nuclear property common to the concepts of round and square. It is an interesting way to represent a non-contradictory round square which is more than the minimal round square.

The round square o_4 is an *incoherent contradictory object*. It is contradictory in a minimal way: the only pair of contradicting nuclear properties are the pair *Round* and *not-Round*.

And finally, the round square o_2 is a *coherent contradictory object*. Since it is coherently *Round*, it is *Round*, *Curved*, *non-Square*, *non-Four-sided*; and since it is coherently *Square*, it is also *Square*, *Four-sided*, *non-Round* and *non-Curved*. Therefore it is plainly contradictory.

If we expect the round square to be a contradictory object, we must thus choose between o_2 and o_4 . If we expect moreover that the round square is a coherent object, then only o_2 is satisfying. (Note that a round square cannot be coherent without being contradictory.)

5. Conclusion. Outline of a Generalization.

A domain of Meinongian objects is sometimes compared to a jungle, and generally it is not a friendly comparison. I think however that the mereological tools I have defined in this paper allow us to explore systematically this realm,

and far from being as chaotic and confusing as the image of a jungle suggests, this ontology presents a robust logical structure.

The results obtained in the theory M could presumably be obtained in similar theories based on a distinction between two kinds of predicates, but one may wonder if equivalent results can also be obtained in other forms of Meinongian theory. For purely mereological results (i.e., results presented in section 3), I think it is the case: equivalent results can be obtained in any framework providing a domain of Meinongian objects. Here, I will only sketch an outline of this generalization.

A domain of Meinongian objects is described by a principle of this form:

(P) *For any class of characteristic properties, there is an object whose characteristic properties are exactly all the member of this class.*

If you do not want to use the concept of class, this principle can be formulated like this: *for any condition* ϕ , there is an object whose characteristic properties are exactly all the properties satisfying ϕ . (It is a principle of this form that I use in theory M).

Meinongian theories can differ in the way they give an account of what is a characteristic property (and therefore in the way the principle (P) must be understood). But whatever is a characteristic property for a Meinongian theory, such a theory allows the construction of a Meinongian mereology as follows.

Let us define a *Meinongian* object as an object described by an instance of (P). In other terms, an object is Meinongian iff an instance of (P) assures that there is such an object.¹⁸

In what follows, x and y are supposed to be Meinongian objects.

We assume the following definitions:

x is a *part* of y iff each characteristic property of x is a characteristic property of y .

x *overlaps* y iff x and y share at least one characteristic property.

x is an atom iff x has a single characteristic property.

The *sum* of x and y is the Meinongian object having every characteristic property of x and every characteristic property of y .

¹⁸ In Parsons' theory, all objects are Meinongian (it simplifies the presentation of the mereology), but this does not hold for other varieties of Meinongianism. For instance, in Zalta's theory, Meinongian objects are only abstract objects. Concrete objects are not Meinongian. Therefore the mereology that can be developed in this theory would only concerns abstract objects. (The results about concepts of properties and their relation with concrete objects would be very different from those I obtained in theory M (in 4.1.)

The *product* of x and y is the Meinongian object having every characteristic property common to x and y .

The *complement* of x is the Meinongian object having every characteristic property x does not have.

The *null object* is the Meinongian object having no characteristic property.

The *total object* is the Meinongian object having all characteristic properties.

The principle (P) should imply that there is indeed a null object and a total object. From those definitions, we should be able to prove, for each theorem of section 3, an equivalent theorem. For instance:

For every characteristic property there is a unique atom having this property.

Two Meinongian non-atomic objects are identical iff they share exactly the same atoms as parts.

Every Meinongian object is the sum of the atoms of its characteristic properties.

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